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On generically tame algebras over perfect fields

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Abstract

Given a generically tame finite-dimensional algebra Λ over an infinite perfect field, we give, for each natural number d, parametrizations of the indecomposable Λ-modules with dimension d similar to those occurring for the algebraically closed field case. We parametrize over bounded principal ideal domains, instead of over rational algebras.

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1. Introduction

An important effort has been made to explore the notion of tameness for general finite-dimensional algebras over non-algebraically closed fields. The particular case of hereditary algebras is quite well understood, after the work of Dlab and Ringel (see [17,18,26]), and also from the work of Crawley-Boevey (see [15]). The last author has pointed out the need to understand first the hereditary case because this will be needed for the comprehension of

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the general situation, he has particularly underlined this for the possible extensions of Drozd's Tame and Wild Theorem; see [19,13]. The results of our paper support this vision since, in the case of generically tame algebras over infinite perfect fields, the study of finite-dimensional indecomposables is almost reduced to the case of tame bimodules (see Theorem 1.1). Our main result appears as Theorem 1.2 below (and in a more complete formulation as Theorem 10.2). This result provides, for any given generically tame finite-dimensional algebra over an infinite perfect field and any natural number d, parametrizations over bounded principal ideal domains for almost every d-dimensional indecomposable module of the given algebra.

Denote by k a fixed ground field and let Λ be any k-algebra (associative, with unit element 1). Given a Λ-module G, recall that, by definition, the *endolength* of G is its length as a right $\mathrm{End}_\Lambda(G)^{op}$-module. The module G is called *generic* iff it is indecomposable, of infinite length as a Λ-module but with finite endolength. The algebra Λ is called *generically tame* iff, for each $d \in \mathbb{N}$, there are only finitely many isomorphism classes of generic Λ-modules of endolength d. This notion was introduced by Crawley-Boevey in [14], providing a new definition of tameness, which coincides with the usual notion of tameness for finite-dimensional algebras over algebraically closed fields, but which makes sense for arbitrary algebras.

Over an algebraically closed field k, there are various characterizations of tameness. Let us have in mind the following one [see [8](27.5)/(27.14)]. The finite-dimensional algebra Λ is tame iff, for each natural number d, there is a finite sequence $\{(B_i, Z_i)\}_{i=1}^m$, where $B_i = k[x]_{f_i}$ is a rational algebra and Z_i is a Λ–B_i-bimodule, which is finitely generated as right B_i-module, such that the functors $Z_i \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserve isomorphism classes of indecomposables, and almost every d-dimensional indecomposable Λ-module M is of the form $M \cong Z_i \otimes_{B_i} N$, for some $i \in [1, m]$ and $N \in B_i\text{-Mod}$.

Here, we say that a functor F *preserves isomorphism classes of indecomposables* iff $F(N)$ is indecomposable whenever N is so, and whenever $F(N) \cong F(N')$ with N and N' indecomposables, we have that $N \cong N'$. Also, when we say that *almost all modules in a class $\mathcal{C}$ of objects in a category satisfy some property*, we mean that every $M \in \mathcal{C}$ has this property, with the possible exception of those M lying in a finite union of isomorphism classes in $\mathcal{C}$.

In a first step, we will prove the following statement, for generically tame finite-dimensional algebras over perfect fields, which provides a family of "parametrizing bimodules", but instead of using modules over rational algebras as parameters, we have to use modules over hereditary algebras B of one of the following three types:

1. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 2 = \dim M_F$; or
2. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 4$ and $\dim M_F = 1$, or $\dim_G M = 1$ and $\dim M_F = 4$; or
3. B is a skew polynomial algebra $D[x, \alpha]$, where D is a finite-dimensional division k-algebra and α is some automorphism of D.

More precisely, we prove the following result, where the word *regular* applied to a $D[x, \alpha]$-module means that it is finite-dimensional.

Theorem 1.1. *Let Λ be a generically tame finite-dimensional algebra over an infinite perfect field k and let d be a non-negative integer. Then, there is a finite sequence $\{(B_i, Z_i)\}_{i=1}^m$, where*

B_i is one of the hereditary algebras listed above and Z_i is a Λ–B_i-bimodule, which is finitely generated as right B_i-module, satisfying the following.

1. *The functor $Z_i \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes of modules without injective direct summands, if B_i is a hereditary algebra of one of the types 1 or 2.*
2. *The functor $Z_i \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes, if B_i is a hereditary algebra of type 3.*
3. *Almost every indecomposable Λ-module M with $\dim_k M \leq d$ is isomorphic to $Z_i \otimes_{B_i} N$, for some $i \in [1, m]$ and some $N \in B_i\text{-mod}$.*
4. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in $B_i\text{-mod}$ and $B_j\text{-mod}$, respectively, such that $Z_i \otimes_{B_i} N_u \cong Z_j \otimes_{B_j} M_u$ for all $u \in U$, then $i = j$.*

The idea of generalized one-parameter families over finite-dimensional algebras was already considered by Simson in [30](Problem 2.11(a)).

Then, in a second step, we proceed to provide parametrizations over principal ideal domains.

Theorem 1.2. *Let Λ be a generically tame finite-dimensional algebra over an infinite perfect field k and let d be a non-negative integer. Then, there is a finite sequence of bounded principal ideal domains $\Gamma_1, \ldots, \Gamma_m$ and Λ–Γ_i-bimodules $\hat{Z}_1, \ldots, \hat{Z}_m$, which are finitely generated as right Γ_i-modules, satisfying the following.*

1. *The functor $\hat{Z}_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes.*
2. *Almost every indecomposable Λ-module M with $\dim_k M \leq d$ is isomorphic to $\hat{Z}_i \otimes_{\Gamma_i} N$, for some $i \in [1, m]$ and some $N \in \Gamma_i\text{-mod}$.*
3. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in $\Gamma_i\text{-mod}$ and $\Gamma_j\text{-mod}$, respectively, such that $\hat{Z}_i \otimes_{\Gamma_i} N_u \cong \hat{Z}_j \otimes_{\Gamma_j} M_u$ for all $u \in U$, then $i = j$.*

The proof of our main result for algebras relies on the theory of differential tensor algebras (*ditalgebras*) and reduction functors first developed by the Kiev School of representation theory of algebras (the same as Drozd's proof of his Tame and Wild Theorem). For the general background on ditalgebras and their module categories, we refer the readers systematically to [8]. We tried to give precise references for the basic terminology and ditalgebra constructions. Should there be some undefined terms, the reader may look for them in the index of [8]. In the following lines we describe the main ingredients of this writing.

After some preparation, we give in (3.5) an adaptation of the main theorem of [5], which describes some remarkable extension/restriction interaction between the module category of an initial subalgebra B of a pregenerically tame ditalgebra $\mathcal{A}$, over an algebraically closed field, and the module category of $\mathcal{A}$. Then, in Sections 4 and 5, using Kasjan's work (see [22,23]), we extend the previous interaction for perfect fields. We consider the scalar extension to the algebraic closure and we use the scalar extension properties for ditalgebras studied in [8].

In Section 6, we introduce what we call *minimal algebras* and, based on previous work of Dlab, Ringel and Crawley-Boevey, we review some basic relations between generically tame minimal algebras and some principal ideal domains.

In Section 7, we present the process of reduction to minimal algebras, which proves a version of Theorem 1.1 for ditalgebras; in Section 8, this is done for finite-dimensional algebras over perfect fields. Section 9 is devoted to the study of preservation of almost split sequences by extension functors. This is applied in the last section to give, in our Theorem 10.2, some description of almost split sequences involving the parametrizing families over the principal ideal domains given by (1.2), which is similar to the situation occurring for tame algebras in the algebraically closed field case (see [14](5.4) and [8](32.11)).

When the field k is finite, our results are trivially true for any finite-dimensional k-algebra Λ. The reasonable problem in the finite field case is to parametrize the indecomposable finite-dimensional Λ-modules with bounded endolength, which presents other technical difficulties. This is an interesting problem which is left for later study.

2. Endolength and pregeneric tameness

As usual, given any k-ditalgebra $\mathcal{A}$, we denote by $\mathcal{A}$-Mod the category of $\mathcal{A}$-modules, as in [8](2.2). The full subcategory of $\mathcal{A}$-Mod formed by the finite-dimensional $\mathcal{A}$-modules is denoted by $\mathcal{A}$-mod.

Definition 2.1. Let $\mathcal{A}$ be a layered ditalgebra, with layer (R, W); see [8]Section 4. Given $M \in \mathcal{A}$-Mod, denote by $E_M := \mathrm{End}_{\mathcal{A}}(M)^{op}$ its endomorphism algebra. Then, M admits a structure of R–E_M-bimodule, where $m \cdot (f^0, f^1) = f^0(m)$, for $m \in M$ and $(f^0, f^1) \in E_M$. By definition, the *endolength* of M, denoted by $\mathrm{endol}\,(M)$, is the length of M as a right E_M-module.

A module $M \in \mathcal{A}$-Mod is called *pregeneric* iff M is indecomposable, with finite endolength but with infinite dimension over the ground field k. The ditalgebra $\mathcal{A}$ is called *pregenerically tame* iff, for each natural number d, there are only finitely many isoclasses of pregeneric $\mathcal{A}$-modules of endolength d.

If B is any k-algebra, we have the corresponding regular ditalgebra $\mathcal{A}$ with layer $(B, 0)$. We identify canonically the categories $\mathcal{A}$-Mod with B-Mod. Thus, if B is a finite-dimensional algebra, the notions of *pregeneric module* and *pregeneric tameness* for the algebra B coincide with the usual notions of *generic module* and *generic tameness*. This is not the case for arbitrary infinite-dimensional k-algebras. For instance, if $B = k(x)[y]$ and $\lambda \in k$, then the B-module $B/\langle y - \lambda\rangle$ is pregeneric but not generic. The use of the notion *pregeneric* enables us to present more precisely some of our arguments.

Lemma 2.2. *Assume that $\xi : \mathcal{A} \longrightarrow \mathcal{A}'$ is a morphism of layered ditalgebras and consider the functor $F_\xi : \mathcal{A}'$-Mod $\longrightarrow \mathcal{A}$-Mod induced by restriction using the morphism ξ. Given $M \in \mathcal{A}'$-Mod, we always have* $\mathrm{endol}\,(F_\xi(M)) \leq \mathrm{endol}\,(M)$. *If F_ξ is full, then we have the equality* $\mathrm{endol}\,(M) = \mathrm{endol}\,(F_\xi(M))$. *Moreover, if F_ξ is an equivalence, then $\mathcal{A}$ is pregenerically tame iff $\mathcal{A}'$ is so.*

Proof. Given any $\mathcal{A}'$-module M, we have the algebras $E_M := \mathrm{End}_{\mathcal{A}'}(M)^{op}$ and $E_{F_\xi(M)} := \mathrm{End}_{\mathcal{A}}(F_\xi(M))^{op}$. The functor F_ξ induces a morphism of algebras $E_M \longrightarrow E_{F_\xi(M)}$ such that $(f^0, f^1) \mapsto (f^0, f^1\xi_1)$. Then, every chain of $E_{F_\xi(M)}$-submodules of $F_\xi(M)$ is a chain of E_M-submodules of M and, therefore, we obtain $\mathrm{endol}\,(F_\xi(M)) \leq \mathrm{endol}\,(M)$. Whenever the functor F_ξ is full, the converse holds and we obtain the equality $\mathrm{endol}\,(M) = \mathrm{endol}\,(F_\xi(M))$. If, moreover, F_ξ is an equivalence, it induces a bijection between the corresponding isoclasses of pregeneric modules of a given endolength. $\square$

Reminder 2.3. Throughout this work, given a ditalgebra $\mathcal{A} = (T, \delta)$, in agreement with [8], we denote with a roman A the subalgebra $[T]_0$ of degree zero elements of the underlying graded algebra T of $\mathcal{A}$; see [8]Section 1. Then, the categories A-Mod and $\mathcal{A}$-Mod share the same class of objects, but there are more morphisms in $\mathcal{A}$-Mod. There is a canonical embedding

$$L_{\mathcal{A}} : A\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$$

which is the identity on objects and $L_{\mathcal{A}}(f^0) = (f^0, 0)$ for any $f^0 \in \mathrm{Hom}_A(M, N)$. The functor $L_{\mathcal{A}}$ is additive and if $\mathcal{A}$ is a Roiter ditalgebra, it maps exact sequences onto exact pairs. In this case, if (R, W) denotes the layer of $\mathcal{A}$, then the images under $L_{\mathcal{A}}$ of the exact sequences in A-Mod which split in R-Mod represent the conflations in the canonical exact structure of $\mathcal{A}$-Mod; see [8]Section 6. This exact structure will be relevant in Section 9.

We recall some terminology from [8,5]. Let $\mathcal{A} = (T, \delta)$ be any ditalgebra with layer (R, W). Assume that we have R–R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. Consider the subalgebra T' of T generated by R and $W' = W_0' \oplus W_1'$, and the subalgebra A' of A generated by R and W_0'. Let us assume furthermore that $\delta(W_0') \subseteq A'W_1'A'$ and $\delta(W_1') \subseteq A'W_1'A'W_1'A'$. Then, the differential δ on T restricts to a differential δ' on the algebra T' and we obtain a new ditalgebra $\mathcal{A}' = (T', \delta')$ with layer (R, W'). A layered ditalgebra $\mathcal{A}'$ is called a *proper subditalgebra of* $\mathcal{A}$ if it is obtained from an R–R-bimodule decomposition of W as we have just described.

A proper subditalgebra $\mathcal{A}'$ of a triangular ditalgebra $\mathcal{A}$ is called *initial* when its triangular filtrations coincide with the first terms of the triangular filtrations of $\mathcal{A}$; see [8](14.8).

The inclusion $r : T' \longrightarrow T$ yields a morphism of ditalgebras $r : \mathcal{A}' \longrightarrow \mathcal{A}$ and, hence, a *restriction functor* (see [8](2.4))

$$R^{\mathcal{A}}_{\mathcal{A}'} := F_r : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}.$$

The projection $\pi : A = [T]_0 \longrightarrow [T']_0 = A'$ yields an *extension functor*

$$E^{A}_{A'} := F_\pi : A'\text{-Mod} \longrightarrow A\text{-Mod}.$$

Since there is no danger of confusion, we forget subindices and superindices in restriction and extension functors.

Let $\mathcal{A} = (T, \delta)$ be a ditalgebra with layer (R, W). Then, an algebra B is called a *proper subalgebra of* $\mathcal{A}$ if and only if $B = [T']_0$, for some proper subditalgebra $\mathcal{A}' = (T', \delta')$ of $\mathcal{A}$ associated to R–R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, where $W_1' = 0$.

Lemma 2.4. *Assume that $\mathcal{A}'$ is a proper subditalgebra of the pregenerically tame layered ditalgebra $\mathcal{A}$. Let $E : A'\text{-Mod} \longrightarrow A\text{-Mod}$ denote the extension functor. Then, we have the following.*

1. *The module $E(M)$ is indecomposable in* $\mathcal{A}$-Mod, *whenever M is indecomposable in* $\mathcal{A}'$-Mod.
2. $E(M) \cong E(N)$ *in* $\mathcal{A}$-Mod *implies that $M \cong N$ in* $\mathcal{A}'$-Mod.
3. *If $\mathcal{A}'$ is a proper subalgebra of $\mathcal{A}$, we have*

$$\mathrm{endol}\,(E(M)) = \mathrm{endol}\,(M), \quad \textit{for any } M \in \mathcal{A}'\text{-Mod}.$$

Thus, the algebra $\mathcal{A}'$ is pregenerically tame, whenever $\mathcal{A}$ is so.

Proof. Denote by $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}$ the restriction functor. Recall that if $\xi : \mathcal{A}' \longrightarrow \mathcal{A}$ denotes the inclusion of ditalgebras and $\pi : A \longrightarrow A'$ denotes the canonical projection of algebras, then $R = F_\xi$ is induced by restriction using ξ and E is induced by restriction using π. It follows that $RE(M) = M$, for any A'-module M. Then, the first two items are clear.

Given $M \in \mathcal{A}'$-Mod, we apply (2.2) to the morphism ξ and to the module $E(M)$, to obtain $\mathrm{endol}\,(M) = \mathrm{endol}\,(RE(M)) \leq \mathrm{endol}\,(E(M))$. If $\mathcal{A}'$ is a subalgebra of $\mathcal{A}$, then the rule $(f^0, 0) \mapsto (f^0, 0)$ makes E a functor $\mathcal{A}'\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ such that $RE = 1_{\mathcal{A}'\text{-Mod}}$. In particular, R is a full functor and we can apply (2.2) to get the equality in 3.

If the subalgebra $\mathcal{A}'$ of $\mathcal{A}$ had an infinite family of pairwise non-isomorphic pregeneric modules with bounded endolength, their image under E would be an infinite family of pairwise non-isomorphic pregeneric $\mathcal{A}$-modules with bounded endolength. $\square$

From (2.2), we immediately obtain the following two lemmas.

Lemma 2.5. *Let $\mathcal{A} = (T, \delta)$ be a ditalgebra with layer (R, W) over a field k. Suppose that $\mathcal{A}^d$ is obtained from $\mathcal{A}$ by deletion of some idempotents of R, as in* [8](8.17). *Then, $\mathcal{A}^d$ is a layered ditalgebra and the associated reduction functor $F_d : \mathcal{A}^d\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is full, faithful and preserves endolength. Therefore, $\mathcal{A}^d$ is pregenerically tame whenever $\mathcal{A}$ is so.*

Lemma 2.6. *Let $\mathcal{A} = (T, \delta)$ be a ditalgebra with layer (R, W) over a field k. Assume that we have the R–R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = \delta(W_0') \oplus W_1''$. Suppose that $\mathcal{A}^r$ is obtained from $\mathcal{A}$ by regularization of the bimodule W_0', as in* [8](8.19). *Then, $\mathcal{A}^r$ is a layered ditalgebra and the associated reduction functor $F_r : \mathcal{A}^r\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is full, faithful and preserves endolength. Therefore, $\mathcal{A}^r$ is pregenerically tame whenever $\mathcal{A}$ is so.*

Lemma 2.7. *Let $\mathcal{A}$ be a layered ditalgebra over a field k. Assume that $\mathcal{A}^X$ is the layered ditalgebra obtained from $\mathcal{A}$ by reduction, using the $\mathcal{B}$-module X, where $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$ and X is a finite direct sum of finite-dimensional indecomposable $\mathcal{B}$-modules; see* [8](12.9). *Assume that X is a complete admissible triangular $\mathcal{B}$-module and consider the associated functor $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, as in* [8](12.10). *Then, for all $N \in \mathcal{A}^X\text{-Mod}$, we have that*

$$\mathrm{endol}\,(F_X(N)) \leq \dim_k X \times \mathrm{endol}\,(N).$$

Moreover, $\mathcal{A}^X$ is pregenerically tame, whenever $\mathcal{A}$ is so.

Proof. Suppose $\mathcal{A}$ has layer (R, W). Assume that $\Gamma = \mathrm{End}_B(X)^{op} = S \oplus P$ is a splitting where P is the radical, thus $\mathcal{A}^X$ has layer (S, W^X). From [8](13.5), we know that F_X is full and faithful. Take $N \in \mathcal{A}^X$-Mod and consider the k-algebra $E := \mathrm{End}_{\mathcal{A}^X}(N)^{op}$. We have the isomorphism $E \cong \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$ induced by F_X, which provides, by restriction, a structure of right E-module on $F_X(N)$. Clearly, $\mathrm{endol}\,(F_X(N)) = \ell_E(F_X(N))$ and, hence, all we have to do is to show that $\ell_E(F_X(N)) \leq \dim_k X \times \ell_E(N)$.

We will adapt the proof of [8](25.7), since we have the $\mathcal{A}^X$–E-bimodule N, with bimodule structure $\alpha_N : E \xrightarrow{Id} \mathrm{End}_{\mathcal{A}^X}(N)^{op}$, and the $\mathcal{A}$–E-bimodule $F_X(N)$, having the $\mathcal{A}$–E-bimodule structure $\alpha_{F_X(N)} : E \longrightarrow \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$, which interest us. Recall that the triangular bimodule X admits a right additive B–S-bimodule filtration $\mathcal{F}(X) :\ 0 = X^0 \subseteq X^1 \subseteq \cdots \subseteq X^{\ell_X} = X$, such that $X^t P \subseteq X^{t-1}$, for all $t \in [1, \ell_X]$. We know that $N \in \mathcal{A}^X$–E-Mod is an S–E-bimodule via $ne := \alpha_N(e)^0(n)$, where $\alpha_N : E \longrightarrow \mathrm{End}_{\mathcal{A}^X}(N)^{op}$ is the given $\mathcal{A}$–E-bimodule structure of $N \in \mathcal{A}^X$–E-Mod, $n \in N$ and $e \in E$.

Thus, each $X^t \otimes_S N$ inherits a natural structure of an R–E-bimodule. Namely, $(x \otimes n) \star e := x \otimes (ne)$, for $x \in X^t$ and $n \in N$. We denote the length of submodules or quotients of these modules with the symbol $\ell_E^{\star}$.

Recall from [8](12.10) and [8](21.3), that for any elements $e \in E$, $n \in N$ and $x \in X$, we have $\left(\alpha_{F_X(N)}(e)\right)^0 [x \otimes n] = F_X(\alpha_N(e))^0[x \otimes n] = x \otimes \alpha_N(e)^0(n) + \sum_\xi x p_\xi \otimes \alpha_N(e)^1(\gamma_\xi)[n]$, where $(p_\xi, \gamma_\xi)_\xi$ is a dual basis for the projective right S-module P.

Consider the structure of R–E-bimodule on $F_X(N) = X \otimes_S N$ determined by the $\mathcal{A}$–E-bimodule $F_X^E(N)$, that is $(x\otimes n)\cdot e = \alpha_{F_X(N)}(e)^0[x\otimes n]$, for $x \in X$ and $n \in N$. From the previous formula for $\alpha_{F_X(N)}(e)^0$ we immediately obtain that each $X^t \otimes_S N$ is an R–E-subbimodule of $F_X(N)$. We write the length of submodules or quotients of these modules with the symbol ℓ_E.

As in the proof of [8](25.7), we show by induction on t that $\ell_E^\star(X^t \otimes_S N) = \ell_E(X^t \otimes_S N)$, for any $t \in [0, \ell_X]$. If we write $d := \dim_k X$, we have that the right S-module X is a quotient of the free module S^d. Then, for $t = \ell_X$, we have $X = X^t$ and

$$\ell_E(F_X(N)) = \ell_E(X \otimes_S N) = \ell_E^\star(X \otimes_S N) \leq \ell_E^\star(S^d \otimes_S N) = d \times \ell_E(N).$$

Finally, if we assume that $\mathcal{A}^X$ is not pregenerically tame, we have an infinite family of pairwise non-isomorphic pregeneric $\mathcal{A}^X$-modules with bounded endolength. Then, applying the full and faithful functor F^X to them, we obtain an infinite family of pairwise non-isomorphic pregeneric $\mathcal{A}$-modules of bounded endolength. Hence, $\mathcal{A}$ is not pregenerically tame. $\square$

We recall the definition of wildness.

Definition 2.8. A ditalgebra $\mathcal{A} = (T, \delta)$ over the field k is *wild* if there is an A–$k\langle x, y\rangle$-bimodule Z, free of finite rank as a right $k\langle x, y\rangle$-module, such that the composition functor

$$k\langle x, y\rangle\text{-Mod} \xrightarrow{Z\otimes_{k\langle x,y\rangle}-} A\text{-Mod} \xrightarrow{L_\mathcal{A}} \mathcal{A}\text{-Mod}$$

preserves isomorphism classes of indecomposables, where $A = [T]_0$ and $L_\mathcal{A}$ is the canonical embedding.

Proposition 2.9. *Any pregenerically tame layered ditalgebra $\mathcal{A}$ is not wild.*

Proof. Consider, for any monic irreducible element $p \in k(x)[y]$, the $k\langle x, y\rangle$-module $H_p := k(x)[y]/\langle p\rangle$, where $k\langle x, y\rangle$ acts by restriction via the epimorphism of algebras

$$k\langle x, y\rangle \longrightarrow k[x, y] \longrightarrow k(x)[y].$$

Then, $\{H_p\}_p$ is an infinite family of pairwise non-isomorphic generic $k\langle x, y\rangle$-modules of endolength 1. Assume that $\mathcal{A}$ is wild and adopt the notation of the previous definition, assuming that the rank of Z as a right $k\langle x, y\rangle$-module is n, and make $F := Z \otimes_{k\langle x,y\rangle} -$. Then, we have an infinite family $\{F(H_p)\}_p$ of pairwise non-isomorphic indecomposable $\mathcal{A}$-modules. We can easily adapt the proof of [8](31.4), to obtain that $\text{endol}\,(F(H_p)) \leq n \times \text{endol}\,(H_p)$. Since Z is free of finite rank as a right $k\langle x, y\rangle$-module, each $Z \otimes_{k\langle x,y\rangle} H_p$ is infinite-dimensional. $\square$

3. Admissibility and restrictions

Definition 3.1. Let $\mathcal{A} = (T, \delta)$ be a triangular ditalgebra, with layer (R, W), over any field k. Then,

1. $\mathcal{A}$ is called *admissible* iff $R \cong D_1 \times \cdots \times D_n$, for some finite-dimensional division k-algebras $D_1, \ldots, D_n$, and W is finitely generated as an R–R-bimodule.
2. $\mathcal{A}$ is called *almost admissible* iff $R \cong M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$, for some finite-dimensional division k-algebras $D_1, \ldots, D_n$ and W is finitely generated as an R–R-bimodule.

Example 3.2. If the field k is perfect, any finite-dimensional k-algebra Λ splits over its radical, thus $\Lambda = S \oplus J$, where J is the radical of Λ. Then, we can consider the Drozd's ditalgebra

$\mathcal{D} = \mathcal{D}^{\Lambda}$ of Λ, as in [8](19.1). It is an almost admissible k-ditalgebra, which is admissible if and only if Λ is basic.

If K is any field extension of the perfect field k and $\mathcal{A}$ is an admissible ditalgebra over k, then the extended ditalgebra $\mathcal{A}^K$ is an almost admissible ditalgebra over K (see [8]Section 20).

Proposition 3.3. *Assume that $\mathcal{A} = (T, \delta)$ is an almost admissible ditalgebra, with layer (R, W), over any field k. Then, there is an admissible ditalgebra $\mathcal{A}^b$ and an equivalence of categories $F^b : \mathcal{A}^b\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Moreover, there are positive integers $c, c' \in \mathbb{N}$ such that, for every $N \in \mathcal{A}^b\text{-Mod}$,*

$$\begin{cases} \dim_k N \leq \dim_k F^b(N) \leq c \times \dim_k N \\ \operatorname{endol}(N) \leq \operatorname{endol}(F^b(N)) \leq c' \times \operatorname{endol}(N). \end{cases}$$

We call $\mathcal{A}^b$ the basification *of $\mathcal{A}$. Thus, $\mathcal{A}^b$ is pregenerically tame iff $\mathcal{A}$ is so.*

If $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$, we can simultaneously basify $\mathcal{A}$ and $\mathcal{A}'$, obtaining that $\mathcal{A}'^b$ is an initial subditalgebra of $\mathcal{A}^b$ and the commutative diagram

$$\begin{tikzcd} \mathcal{A}^b\text{-Mod} \arrow[r, "F^b"] \arrow[d, "R^b"'] & \mathcal{A}\text{-Mod} \arrow[d, "R"] \\ \mathcal{A}'^b\text{-Mod} \arrow[r, "F'^b"] & \mathcal{A}'\text{-Mod} \end{tikzcd}$$

where R^b, R denote restriction functors and F^b, F'^b are the corresponding equivalence functors. Moreover, $F^b E^b(M) = E F'^b(M)$, for any $M \in \mathcal{A}'^b\text{-Mod}$, where $E^b : A'^b\text{-Mod} \longrightarrow A^b\text{-Mod}$ and $E : A'\text{-Mod} \longrightarrow A\text{-Mod}$ are the corresponding extension functors.

Proof. Adopt the notation of the second item in the last definition. Choose an indecomposable projective $M_{m_i}(D_i)$-module X_i, for each $i \in [1, n]$. Then we have a finite family $X_1, \ldots, X_n$ of pairwise non-isomorphic finite-dimensional indecomposable R-modules, and any finite-dimensional indecomposable projective R-module is isomorphic to one of them. Consider the R-module $X := X_1 \oplus \cdots \oplus X_n$.

We identify the initial subditalgebra $\mathcal{B}$ with layer $(R, 0)$ of $\mathcal{A}$ with its underlying t-algebra R. Denote by e_i the primitive idempotent corresponding to the direct summand $X_i \cong Re_i$ of R, and make $e = \sum_{i=1}^n e_i$. Then, $X \cong Re$ and $\Gamma = \operatorname{End}_R(X)^{op} \cong eRe$ has zero radical $P = e\operatorname{rad}(R)e = 0$. Thus, $\Gamma = S \oplus P$ splits over its radical $P = 0$. The algebra S is basic because the indecomposable direct summands of X are non-isomorphic. From [8](5.6), we know that $\mathcal{B}$ is a Roiter ditalgebra. Since $\operatorname{End}_R(X_i)^{op} \cong e_i Re_i \cong D_i$ is a division algebra for each $i \in [1, n]$, from [8](17.1)–(17.2), the $\mathcal{B}$-module X is admissible. The $\mathcal{B}$-module X is complete by [8](13.3), see [8](12.4), and it is triangular by [8](17.4). Then, the ditalgebra $\mathcal{A}^X$ is triangular, with layer (S, W^X), its natural triangular structure is described in [8](14.10). Hence, $\mathcal{A}^b := \mathcal{A}^X$ is an admissible ditalgebra.

From [8](13.5), we know that $F^b := F_X : \mathcal{A}^b\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is full and faithful. There is an isomorphism of R-modules $X \otimes_S \operatorname{Hom}_R(Re, M) \cong M$, for each $M \in R\text{-Mod}$; see [1](6.10). It follows, from [8](25.5) that F^b is a dense functor.

Consider the canonical decomposition $1_X = f_1 + \cdots + f_n$ of the unit of Γ as a sum of primitive orthogonal idempotents; thus f_i is the composition $X \xrightarrow{\pi_i} X_i \xrightarrow{\sigma_i} X$ of the corresponding injection and projection, for $i \in [1, n]$. Then, for any given $N \in \mathcal{A}^b\text{-Mod}$, we have isomorphisms of R-modules $F^b(N) = X \otimes_S N \cong \oplus_{i=1}^n Xf_i \otimes_{Sf_i} f_i N \cong \oplus_{i=1}^n (f_i N)^{d_i}$, where d_i denotes the dimension of Xf_i over the division algebra Sf_i, for each $i \in [1, n]$. Notice that $Xf_i = X_i \neq 0$, for all i, thus $d_i \neq 0$, and take $c := \max\{d_1, \ldots, d_n\}$. Then, $\dim_k N \leq \dim_k F^b(N) = \sum_i d_i \dim_k f_i N \leq c \sum_i \dim_k f_i N = c \dim_k N$.

The algebra $E = \mathrm{End}_{\mathcal{A}^X}(N)^{op}$ acts on N by $ne = e^0(n)$, for $n \in N$ and $e \in E$. The same algebra E acts on $F^X(N)$ by restriction through the isomorphism $E \longrightarrow \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$ induced by the functor F_X. Since $P = 0$, we have $F_X(e)^0 = 1_X \otimes e^0$, and this last action is given by $(x \otimes n)e = x \otimes e^0(n) = x \otimes ne$, for $x \in X$, $n \in N$ and $e \in E$. Then, $\mathrm{endol}\,(F_X(N)) = \ell_E(F_X(N)) = \ell_E(X \otimes_S N) = \ell_E(\oplus_j Xf_j \otimes_{Sf_j} f_j N) = \sum_j d_j \ell_E(f_j N) \geq \ell_E(N) = \mathrm{endol}\,(N)$. We can use this fact and (2.7) to obtain the inequalities involving endolength in the statement of our proposition.

For the proof of the last statement of our proposition, see [5](3.5) and [5](3.10). □

Lemma 3.4. *Every admissible ditalgebra $\mathcal{A}$, over an algebraically closed field k, is a nested ditalgebra, as in* [8](23.5).

Proof. We have that the layer (R, W) of $\mathcal{A}$ is such that $R \cong D_1 \times \cdots \times D_n$, for some finite-dimensional division k-algebras. Since, k is algebraically closed, $D_i = k$, for all $i \in [1, n]$. Thus, $R \cong k \times \cdots \times k$: a product of n copies of k. Consider the decomposition $1 = \sum_{i=1}^n e_i$ of the unit of R as a sum of central primitive orthogonal idempotents.

Let us recall an argument of [8](15.7), which shows that any R–R-bimodule V is of the form $V \cong \oplus_{\alpha \in \mathbb{A}} Re_{t(\alpha)} \otimes_k e_{s(\alpha)} R$, where $\mathbb{A}$ is a set and $\alpha \mapsto (s(\alpha), t(\alpha))$ is the receipt of some map $\mathbb{A} \to [1, n]^2$. Indeed, V is a left $R \otimes_k R^{op}$-module, and $R \otimes_k R^{op}$ is a semisimple finite-dimensional k-algebra, which decomposes as $R \otimes_k R^{op} = \oplus_{i,j} Re_i \otimes_k e_j R^{op}$. Since, $Re_i \cong k \cong e_j R^{op}$, the simple $R \otimes_k R^{op}$-modules are of the form $Re_i \otimes_k e_j R^{op}$.

Then, we can apply this argument separately to each term of the triangular filtrations of W_0 and W_1, to obtain that they are all freely generated by finite filtered directed subsets. Thus, $\mathcal{A}$ is a nested ditalgebra. □

Theorem 3.5. *Assume that $\mathcal{A}'$ is an initial subditalgebra of the pregenerically tame almost admissible ditalgebra $\mathcal{A}$, over the algebraically closed field k. Consider the extension functor $E : A'$-Mod $\longrightarrow A$-Mod and the restriction functor $R : \mathcal{A}$-Mod $\longrightarrow \mathcal{A}'$-Mod. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{A}'$-modules such that, for any indecomposable $\mathcal{A}$-module M with* $\dim_k M \leq d$ *and $M \not\cong E(N)$ in $\mathcal{A}$*-Mod*, for any $N \in \mathcal{A}'$*-Mod*, the module $R(M)$ is isomorphic in $\mathcal{A}'$*-Mod *to a direct sum of modules of $\mathcal{I}(d)$.*

Proof. Consider the basifications $\mathcal{A}^b$ and $\mathcal{A}'^b$ of $\mathcal{A}$ and $\mathcal{A}'$, respectively, and adopt the notations of (3.3). From (3.4), the ditalgebras $\mathcal{A}^b$ and $\mathcal{A}'^b$ are seminested and $\mathcal{A}'^b$ is an initial subditalgebra of $\mathcal{A}^b$. Moreover, $\mathcal{A}^b$ is also pregenerically tame, and hence, from (2.9), it is a non-wild seminested ditalgebra. Hence, from Drozd's theorem, $\mathcal{A}^b$ is a tame seminested ditalgebra; see [8](27.10). Then, we can apply [5](4.1). Given $d \in \mathbb{N}$, there exists a finite family $\mathcal{I}^b(d)$ of indecomposable $\mathcal{A}'^b$-modules such that, for any indecomposable $\mathcal{A}^b$-module N with $\dim_k N \leq d$ and $N \not\cong E^b(N')$, for any $N' \in \mathcal{A}'^b$-Mod, then $R^b(N)$ is isomorphic in $\mathcal{A}'^b$-Mod to a direct sum of modules in $\mathcal{I}^b(d)$.

Consider the finite family $\mathcal{I}(d)$ consisting of the indecomposable $\mathcal{A}'$-modules of the form $F'^b(L)$, with $L \in \mathcal{I}^b(d)$. Take any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$ and $M \not\cong E(M')$, for any $M' \in \mathcal{A}'$-Mod. Consider $N \in \mathcal{A}^b$-Mod such that $F^b(N) \cong M$. Then, $\dim_k N \leq \dim_k F^b(N) = \dim_k M \leq d$. From (3.3), we see that $N \not\cong E^b(N')$, for any $N' \in \mathcal{A}'^b$-Mod. Thus, $R^b(N) \cong \oplus_i L_i$, for some $L_i \in \mathcal{I}^b(d)$, and also $R(M) \cong RF^b(N) \cong F'^b R^b(N) \cong \oplus_i F'^b(L_i)$. □

Now, we show a couple of examples of a pregenerically tame ditalgebra $\mathcal{A}$ and an initial subditalgebra $\mathcal{A}'$ where our theorem applies.

Example 3.6. Consider the k-algebra $R := k \times k \times k \times k$ and the corresponding decomposition of the unit of R as a sum of primitive central orthogonal idempotents $1 = e_1 + \cdots + e_4$. Consider the R–R-bimodule $R \otimes_k R$ and its elements $\alpha := e_4 \otimes e_1$, $\beta := e_2 \otimes e_4$, $\hat{\alpha} := e_4 \otimes e_3$, $\hat{\beta} := e_3 \otimes e_4$, $x := e_1 \otimes e_3$ and $y := e_3 \otimes e_2$. Then, we have the R–R-bimodules $W_0 = k\alpha \oplus k\beta \oplus k\hat{\alpha} \oplus k\hat{\beta}$, $W_1 = kx \oplus ky$ and $W := W_0 \oplus W_1$. Consider the tensor algebra $T := T_R(W)$ and the R–R-bimodule morphism $\delta : W \longrightarrow T$ defined by $\delta(\alpha) = 0$, $\delta(\beta) = 0$, $\delta(\hat{\alpha}) = \alpha x$, $\delta(\hat{\beta}) = y\beta$, $\delta(x) = 0$ and $\delta(y) = 0$. We can extend this map to a differential $\delta : T \longrightarrow T$, using [8](4.4). Then, we obtain a ditalgebra $\mathcal{A} = (T, \delta)$ which admits a triangular layer (R, W), with triangular filtrations of W_0 and W_1, respectively, given by

$$0 \subseteq k\alpha \subseteq k\alpha \oplus k\beta \subseteq k\alpha \oplus k\beta \oplus k\hat{\alpha} \subseteq k\alpha \oplus k\beta \oplus k\hat{\alpha} \oplus k\hat{\beta} = W_0$$

and

$$0 \subseteq kx \subseteq kx \oplus ky = W_1.$$

Consider the R–R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$, given by $W_0' = k\alpha \oplus k\beta \oplus k\hat{\alpha}$, $W_0'' = k\hat{\beta}$, $W_1' = kx$ and $W_1'' = ky$. Then, we can look at the proper subditalgebra $\mathcal{A}'$ of $\mathcal{A}$ determined by the given R–R-bimodule decompositions. That is $\mathcal{A}' = (T', \delta')$, where T' is the subalgebra of T generated by R and $W' = W_0' \oplus W_1'$, and δ' is the restriction of δ. Then, the ditalgebra $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$. It has the triangular layer (R, W') with triangular filtrations

$$0 \subseteq k\alpha \subseteq k\alpha \oplus k\beta \subseteq k\alpha \oplus k\beta \oplus k\hat{\alpha} = W_0' \quad \text{and} \quad 0 \subseteq kx = W_1'.$$

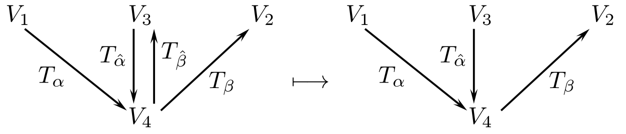
Thus the structure of the layer of $\mathcal{A}'$ is inherited from the structure of the layer of $\mathcal{A}$. In fact, $\mathcal{A}$ is a seminested ditalgebra, and so is $\mathcal{A}'$. The corresponding attached bigraphs $\mathbb{B}$ and $\mathbb{B}'$, see [8](23.9), are the following:

$$\begin{tikzcd}
1 \arrow[dr, "\alpha"'] & 3 \arrow[l, dashed, "x"'] \arrow[d, shift right, "\hat{\alpha}"'] & 2 \arrow[l, dashed, "y"'] \\
\mathbb{B} & 4 \arrow[u, shift right, "\hat{\beta}"'] \arrow[ur, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[dr, "\alpha"'] & 3 \arrow[l, dashed, "x"'] \arrow[d, "\hat{\alpha}"'] & 2 \\
 & 4 \arrow[ur, "\beta"'] & \mathbb{B}'
\end{tikzcd}$$

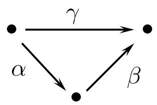
Moreover, we have the extension functor $E^A_{A'} : A'\text{-Mod} \longrightarrow A\text{-Mod}$, where A is the path algebra of the quiver Q obtained from $\mathbb{B}$ after the deletion of the dashed arrows x and y, and A' is the path algebra of the quiver Q' obtained from $\mathbb{B}'$ forgetting the dashed arrow x. The action of this extension functor on any A'-module, that is on any representation of Q', is the following:

$$\begin{tikzcd}
V_1 \arrow[dr, "T_\alpha"'] & V_3 \arrow[d, "T_{\hat{\alpha}}"'] & V_2 \\
 & V_4 \arrow[ur, "T_\beta"'] &
\end{tikzcd}
\longmapsto
\begin{tikzcd}
V_1 \arrow[dr, "T_\alpha"'] & V_3 \arrow[d, shift right, "T_{\hat{\alpha}}"'] & V_2 \\
 & V_4 \arrow[u, shift right, "0"'] \arrow[ur, "T_\beta"'] &
\end{tikzcd}$$

The restriction functor $R^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}$ on any $\mathcal{A}$-module, that is on any A-module or representation of the quiver Q, is the following:

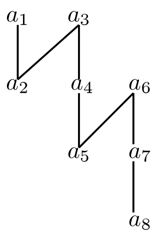


In fact, if we consider the finite-dimensional tame algebra Λ given by the following quiver



as a ditalgebra $\mathcal{B}$ with trivial differential and layer (S, U) with $S = k \times k \times k$, $U_0 = k\alpha \oplus k\beta \oplus k\gamma$ and $U_1 = 0$. Then, $B = \Lambda$ and $\mathcal{B}$-Mod can be naturally identified with Λ-Mod. Then, the ditalgebra $\mathcal{A}$ is obtained from $\mathcal{B}$ by reduction of the edge γ, as described in [8](23.18). Therefore, $\mathcal{A}$ is pregenerically tame. In this case, the conclusion of the theorem is quite trivial because $\mathcal{A}'$ has finite representation type.

Example 3.7. Let $\mathbb{S}$ be the partially ordered set with underlying set $X = \{a_1, \dots, a_8\}$ and partial order defined by the following graph



Consider the ditalgebra $\mathcal{A} = \mathcal{A}^{\mathbb{S}}$ associated to the poset $\mathbb{S}$, as in [8](34.1) and [8](34.2). Then, $\mathcal{A}^{\mathbb{S}}$ is the seminested ditalgebra associated to the bigraph $\mathbb{B}$: with set of points $X \cup \{b\}$, solid arrows $\alpha_i : a_i \to b$, for each $a_i \in X$, and dashed arrows $v_{i,j} : a_j \dashrightarrow a_i$ iff $a_i < a_j$, and differential δ defined by

$$\delta(\alpha_j) = -\sum_{a_i < a_j} \alpha_i v_{i,j} \quad \text{and} \quad \delta(v_{i,j}) = \sum_{a_i < a_t < a_j} v_{i,t} v_{t,j}.$$

For instance, we have $\delta(\alpha_3) = -\alpha_2 v_{2,3} - \alpha_4 v_{4,3} - \alpha_5 v_{5,3}$ and $\delta(v_{5,3}) = v_{5,4} v_{4,3}$. Then, we can consider very natural triangular filtrations for $\mathcal{A}$ in such a way that the following ditalgebra $\mathcal{A}'$ of $\mathcal{A}$ appears as an initial subditalgebra. The seminested ditalgebra $\mathcal{A}'$ has bigraph $\mathbb{B}'$ with the same points of $\mathbb{B}$, and is obtained from the last one by forgetting the solid arrows α_3 and α_6, as well as the dashed arrows $v_{2,3}$, $v_{4,3}$, $v_{5,3}$, $v_{5,6}$, $v_{7,6}$ and $v_{8,6}$.

Consider the subset $X' = \{a_1, a_2, a_4, a_5, a_7, a_8\}$ of X and denote by $\mathbb{S}'$ the full subposet of $\mathbb{S}$ with underlying set X'. Then, the ditalgebra $\mathcal{A}^{\mathbb{S}'}$ associated to the poset $\mathbb{S}'$ is obtained from $\mathcal{A}'$ by deletion of the points a_3 and a_6, and we have the corresponding reduction functor $F^d : \mathcal{A}^{\mathbb{S}'}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}$.

According to Zavadskij's theorem, see [31](15.75), the posets $\mathbb{S}$ and $\mathbb{S}'$ are both tame and with only one one-parameter family of indecomposables for each dimension vector. From [8](34.7), we know that the category of representations of the poset $\mathbb{S}'$ is representation equivalent to $\mathcal{A}^{\mathbb{S}'}$-Mod. Thus, the last one has an infinite family $\{M_\lambda\}_\lambda$ of pairwise non-isomorphic indecomposables with bounded dimension. Then, the full and faithful functor F^d maps this family onto the family $\{F^d(M_\lambda)\}_\lambda$ of pairwise non-isomorphic indecomposable $\mathcal{A}'$-modules with bounded dimension. It is also a family of pairwise non-isomorphic A'-modules. Applying the extension functor $E^A_{A'} : A'\text{-Mod} \longrightarrow A\text{-Mod}$ gives the infinite family $\{E^A_{A'}(F^d(M_\lambda))\}_\lambda$ of A-modules. This is a family of pairwise non-isomorphic indecomposable $\mathcal{A}$-modules because $R^{\mathcal{A}}_{\mathcal{A}'}E^A_{A'}(F^d(M_\lambda)) = F^d(M_\lambda)$ and the restriction functor $R^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}$ is additive. Thus, we have an infinite family $\{E^A_{A'}(F^d(M_\lambda))\}_\lambda$ of pairwise non-isomorphic indecomposable $\mathcal{A}$-modules with bounded dimension which are extended from $\mathcal{A}'$-modules. Here, we have that $\mathcal{A}$ is tame and, hence, it is pregenerically tame; see [6](2.8).

4. Scalar extension and generic tameness

Let us recall some usual notation.

Notation 4.1. Given a finite-dimensional algebra Λ over any field k, denote by $\mathcal{P}(\Lambda)$ the category of morphisms between projective Λ-modules. If we write $J := \operatorname{rad}\Lambda$, then $\mathcal{P}^1(\Lambda)$ denotes the full subcategory of $\mathcal{P}(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with image contained in JQ, and $\mathcal{P}^2(\Lambda)$ denotes the full subcategory of $\mathcal{P}^1(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with kernel contained in JP. If Λ splits over its radical, we can consider Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$, as in [8](19.1), and the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$; see [8](19.8).

Definition 4.2. We say that an almost admissible ditalgebra $\mathcal{A}$, over a perfect field k, is *constructible* iff there is a finite sequence of reductions

$$\mathcal{D}^\Lambda = \mathcal{D} \longmapsto \mathcal{D}^{z_1} \longmapsto \mathcal{D}^{z_1z_2} \longmapsto \cdots \longmapsto \mathcal{D}^{z_1\cdots z_t},$$

where $\mathcal{D}^\Lambda$ is Drozd's ditalgebra of some finite-dimensional k-algebra Λ and there is an isomorphism of layered ditalgebras $\mathcal{D}^{z_1\cdots z_t} \cong \mathcal{A}$, for some finite set of reductions $\mathcal{D}^{z_1\cdots z_{i-1}} \longmapsto \mathcal{D}^{z_1\cdots z_i}$ of either of the types described in (2.5) or in (2.6) or in (2.7). In this case, we also say that $\mathcal{A}$ *is constructible from* Λ.

In this paper, we will mainly consider admissible ditalgebras which are constructible from basic algebras. But sometimes, it is convenient to work with non-basic algebras, because of the following.

Lemma 4.3. *Let k be a perfect field and assume that an almost admissible k-ditalgebra $\mathcal{A}$ is constructible from the finite-dimensional algebra Λ. Then, for any field extension K of k, the K-ditalgebra $\mathcal{A}^K$ is constructible from the finite-dimensional algebra Λ^K.*

Proof. Adopt the notation of last Definition 4.2. Then, our statement follows from the existence of a chain of isomorphisms of layered ditalgebras

$$\mathcal{A}^K \cong \mathcal{D}^{z_1\cdots z_t K} \cong \mathcal{D}^{z_1\cdots z_{t-1}Kz_t} \cong \cdots \cong \mathcal{D}^{z_1Kz_2\cdots z_t} \cong \mathcal{D}^{Kz_1\cdots z_t} \cong (\mathcal{D}^{\Lambda^K})^{z_1\cdots z_t}.$$

Let us be more precise. Since k is perfect, Drozd's ditalgebra $\mathcal{D}^{\Lambda^K}$ of the algebra Λ^K can be identified with $\mathcal{D}^K$ through an isomorphism of layered ditalgebras; see [8](20.13). Once this identification has been made, it is clear that the same sequence of reductions (of type $z_1, \ldots, z_t$) produce isomorphic layered ditalgebras. This was schematized by the last isomorphism of the sequence displayed above. By assumption, there is an isomorphism of layered ditalgebras $\xi : \mathcal{A} \longrightarrow \mathcal{D}^{z_1\cdots z_t}$, which induces the isomorphism of layered ditalgebras $\xi^K : \mathcal{A}^K \longrightarrow \mathcal{D}^{z_1\cdots z_t K}$. This corresponds to the first isomorphism of the displayed sequence.

For each one of the other isomorphisms, we have to look separately at the possible type of reduction at each step.

If, at step i, we consider a reduction by an admissible module X_i, denote by $\mathcal{B}^{z_1\cdots z_{i-1}}$ the initial subalgebra of $\mathcal{D}^{z_1\cdots z_{i-1}}$, which is used to perform the corresponding reduction. Thus, $X_i \in \mathcal{B}^{z_1\cdots z_{i-1}}$-mod. In such case, from [8](20.9), the module X_i extends to K and, from [8](20.11), X_i^K is an admissible $\mathcal{B}^{z_1\ldots z_{i-1}K}$-module, where $\mathcal{B}^{z_1\ldots z_{i-1}K}$ is an initial subalgebra of $\mathcal{D}^{z_1\ldots z_{i-1}K}$. Moreover, there is an isomorphism $\xi_i : \mathcal{D}^{z_1\cdots z_{i-1}K X_i^K} \longrightarrow \mathcal{D}^{z_1\cdots z_{i-1}z_i K}$ of layered ditalgebras. Once again, if we identify these layered ditalgebras, the same sequence of reductions (of type $z_{i+1},\ldots,z_t$) produce isomorphic layered ditalgebras.

Now, in the case that the reduction at step i is a regularization as in (2.6), from [8](20.5), there is an isomorphism $\xi_i : \mathcal{D}^{z_1\cdots z_{i-1}K r^K} \longrightarrow \mathcal{D}^{z_1\cdots z_{i-1}z_i K}$ of layered ditalgebras. Here, the regularization denoted by r^K is determined by the R^K-bimodule decompositions $W_0^K = W_0'^K \oplus W_0''^K$ and $W_1^K = \delta^K(W_0'^K) \oplus W_1''^K$.

The case where the reduction at step i is a deletion of idempotents, as in (2.5), is treated in a similar way, now using [8](20.4). $\square$

Lemma 4.4. *Given a finite-dimensional algebra Λ, over any field k, which splits over its radical, consider the Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^{\Lambda}$, the equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$ and the cokernel functor* Cok $: \mathcal{P}^1(\Lambda) \longrightarrow \Lambda$-Mod. *Assume that $N \in \mathcal{D}$-Mod and $M \in \Lambda$-Mod are such that $\Xi_\Lambda(N) \in \mathcal{P}^2(\Lambda)$ and $M \cong \mathrm{Cok}\,\Xi_\Lambda(N)$. Then, we have the following inequalities:*

1. $\dim_k M \leq \dim_k \Lambda \times \dim_k N$
2. $\dim_k N \leq (1 + \dim_k \Lambda) \dim_k \Lambda \times \dim_k M$
3. $\mathrm{endol}\,(N) \leq (1 + \dim_k \Lambda) \times \mathrm{endol}\,(M)$
4. $\mathrm{endol}\,(M) \leq \dim_k \Lambda \times \mathrm{endol}\,(N)$.

Proof. We may assume that $M = \mathrm{Cok}\,\Xi_\Lambda(N)$. Suppose that $X := \Xi_\Lambda(N) = (P, Q, \phi) \in \mathcal{P}^2(\Lambda)$, then we have a minimal projective presentation of M

$$P \xrightarrow{\phi} Q \longrightarrow M \longrightarrow 0.$$

Assume moreover that $\Lambda = S \oplus J$ is the splitting of Λ over its radical J.
(1) From [8](22.19), we have $\dim_k P/JP + \dim_k Q/JQ = \dim_k N$. Then, as in the proof of [8](29.5), one shows that

$$\dim_k M \leq \dim_k Q \leq \dim_k \Lambda \times \dim_k Q/JQ \leq \dim_k \Lambda \times \dim_k N.$$

(2) From [8](27.13), we know that $\ell_\Lambda(P/JP) \leq \dim_k \Lambda \times \dim_k M$ and also that $\ell_\Lambda(Q/JQ) \leq \dim_k M$. Then, from [8](22.19), we obtain

$$\begin{aligned} \dim_k N &= \dim_k(P/JP) + \dim_k(Q/JQ) \\ &\leq (\dim_k \Lambda)\ell_S(P/JP) + (\dim_k \Lambda)\ell_S(Q/JQ) \\ &\leq (\dim_k \Lambda)[\ell_\Lambda(P/JP) + \ell_\Lambda(Q/JQ)] \\ &\leq (\dim_k \Lambda)[\dim_k \Lambda \times \dim_k M + \dim_k M]. \end{aligned}$$

(3) Consider the endomorphism algebras $E := \mathrm{End}_{\mathcal{P}(\Lambda)}(X)^{op}$, $E' := \mathrm{End}_{\mathcal{D}}(N)^{op}$ and $\overline{E} := \mathrm{End}_\Lambda(M)^{op}$. Then, we have the $\mathcal{D}$–E'-bimodule $\underline{N} \in \mathcal{D}$–$E'$-Mod with bimodule structure

given by the algebra morphism $Id : E' \longrightarrow E'$; by restriction, we have the $\mathcal{D}$–E-bimodule $\overline{N} \in \mathcal{D}$–$E$-Mod with bimodule structure given by the algebra morphism $\Xi_\Lambda^{-1} : E \longrightarrow E'$. Then, the functor $\Xi_\Lambda^E : \mathcal{D}$–$E$-Mod $\longrightarrow \mathcal{P}^1(\Lambda)^E$ maps $\overline{N}$ onto $\Xi_\Lambda^E(\overline{N}) = \underline{X}$, where $\underline{X}$ denotes the object X equipped with the bimodule structure map $Id : E \longrightarrow E$, see [8](21.9). Then, from [8](21.10) and [8](29.5), we have

$$\operatorname{endol}(N) = \ell_{E'}(N) = \ell_E(N) = \ell_E(P/JP) + \ell_E(Q/JQ) \leq (1 + \dim_k \Lambda) \times \ell_E(M).$$

Since $\overline{E}$ is a quotient of E, we also have that $\ell_E(M) \leq \ell_{\overline{E}}(M) = \operatorname{endol}(M)$, and we have the third inequality.
(4) With the notation fixed above, assume that $\operatorname{endol}(N)$ is finite. From the above arguments, $\ell_E(Q/JQ)$ and $\ell_E(P/JP)$ are finite. Then, look at the proof of [8](29.5), where it is shown that $s := \ell_E(Q) \leq \dim_k \Lambda \times \ell_E(Q/JQ)$ (and, similarly, we have that $\ell_E(P) \leq \dim_k \Lambda \times \ell_E(P/JP)$). If we consider the projection $\pi : Q \longrightarrow M$ and a composition series $0 \subseteq Q_s \subseteq \cdots \subseteq Q_1 \subseteq Q_0 = Q$ of the right E-module Q, then we obtain an E-module filtration $0 \subseteq \pi(Q_s) \subseteq \cdots \subseteq \pi(Q_1) \subseteq \pi(Q_0) = M$ such that every factor is either zero or a simple E-module. Hence, $\ell_E(M) \leq \ell_E(Q)$. The last filtration is also a filtration of $\overline{E}$-modules, hence $\operatorname{endol}(M) = \ell_{\overline{E}}(M) \leq \ell_E(Q) \leq \ell_E(P) + \ell_E(Q)$. Thus,

$$\operatorname{endol}(M) \leq \dim_k \Lambda \times [\ell_E(P/JP) + \ell_E(Q/JQ)] = \dim_k \Lambda \times \operatorname{endol}(N). \quad \square$$

Corollary 4.5. *Let Λ be a finite-dimensional algebra over any field k, which splits over its radical. Then, the algebra Λ is generically tame iff its Drozd's ditalgebra $\mathcal{D}$ is pregenerically tame.*

Proof. Assume first that $\mathcal{D}$ is not pregenerically tame, then there is a positive number $d \in \mathbb{N}$ and an infinite family of pairwise non-isomorphic pregeneric $\mathcal{D}$-modules $\{G_i\}_{i\in I}$ with endolength $\leq d$. Notice that there are only finitely many indecomposable isoclasses in $\mathcal{P}^1(\Lambda) \setminus \mathcal{P}^2(\Lambda)$, those represented by objects of the form $(P, 0, 0)$ where P is an indecomposable projective Λ-module; see [8](18.9). For $i \in I$, the pregeneric $\mathcal{D}$-module G_i is infinite-dimensional, then $\Xi_\Lambda(G_i) \in \mathcal{P}^2(\Lambda)$. Indeed, the indecomposable object $\Xi_\Lambda(G_i)$ cannot be isomorphic to $(P, 0, 0)$ because either the domain or the codomain of $\Xi_\Lambda(G_i)$ is infinite-dimensional. From the fourth inequality of last lemma, we know that, for each one of them, we have $\operatorname{endol}(\operatorname{Cok}\Xi_\Lambda(G_i)) \leq \dim_k \Lambda \times \operatorname{endol}(G_i) \leq d \times \dim_k \Lambda$. Then, Λ is not generically tame, because there is an infinite family $\{\operatorname{Cok}\Xi_\Lambda(G_i)\}_{i\in I}$ of pairwise non-isomorphic generic Λ-modules with bounded endolength.

Now, assume that Λ is not generically tame and consider an integer $d \in \mathbb{N}$ and a family of pairwise non-isomorphic generic Λ-modules $\{G_i\}_{i\in I}$ with endolength bounded by d. Choose indecomposable $\mathcal{D}$-modules $\{H_i\}_{i\in I}$ in $\mathcal{D}$-Mod such that $\operatorname{Cok}\Xi_\Lambda(H_i) \cong G_i$. Then, from the third inequality of the last lemma, we obtain that each H_i is a pregeneric $\mathcal{D}$-module with endolength bounded by $(1 + \dim_k \Lambda) \times d$. Hence, $\mathcal{D}$ is not pregenerically tame. $\square$

Proposition 4.6. *Let k be a perfect field and assume that $\mathcal{A}$ is an almost admissible ditalgebra, which is constructible from the generically tame (not necessarily basic) finite-dimensional algebra Λ. Then, $\mathcal{A}$ is pregenerically tame.*

Proof. Assume that $\mathcal{A}$ is not pregenerically tame and let us prove that Λ is not generically tame. From (4.5), it will be enough to show that $\mathcal{D} = \mathcal{D}^\Lambda$ is not pregenerically tame.

Adopt the notation of last Definition 4.2 and consider an isomorphism $\xi : \mathcal{A} \longrightarrow \mathcal{D}^{z_1 \cdots z_t}$ of layered ditalgebras. It induces an equivalence of categories F_ξ and, from (2.2), we know that $\mathcal{D}^{z_1 \cdots z_t}$ is not pregenerically tame. Now our claim follows by induction. At the step i, assume that $\mathcal{D}^{z_1 \cdots z_i}$ is not pregenerically tame. We have three possible types of reduction, for each one of them, using (2.5), (2.6) or (2.7), we get that $\mathcal{D}^{z_1 \cdots z_{i-1}}$ is not pregenerically tame. □

Theorem 4.7. *Let k be a perfect field and denote by K its algebraic closure. Let $\mathcal{A}$ be an almost admissible k-ditalgebra, which is constructible from the generically tame (not necessarily basic) finite-dimensional algebra Λ. Then, the K-ditalgebra $\mathcal{A}^K$ is pregenerically tame.*

Proof. It is known that a finite-dimensional generically tame algebra Λ over a perfect field k, induces a generically tame algebra Λ^K; see [23](5.2) and [25]. From (4.3) and (4.6), the K-ditalgebra $\mathcal{A}^K$ is pregenerically tame. □

5. Restrictions over perfect fields

Lemma 5.1. *Let $\mathcal{A}$ be an admissible ditalgebra, over any field k, and take any field extension K of k. Consider the scalar extension functor $(-)^K : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}^K\text{-Mod}$, as in* [8](20.2). *Assume that $M, N \in \mathcal{A}\text{-mod}$ satisfy that M^K and N^K have a common non-zero direct summand. Then, whenever M is indecomposable, we have that M is a direct summand of N.*

Proof. Given the finite-dimensional $\mathcal{A}$-modules M and N, the natural map

$$\alpha : \mathrm{Hom}_{\mathcal{A}}(M, N)^K \longrightarrow \mathrm{Hom}_{\mathcal{A}^K}(M^K, N^K),$$

is an isomorphism. Indeed, from [8](6.13), we have a commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(M, N)^K & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathcal{A}^K}(M^K, N^K) \\
{\scriptstyle \sigma_{\mathcal{A}} \otimes 1}\Big\downarrow & & {\scriptstyle \sigma_{\mathcal{A}^K}}\Big\downarrow \\
\mathrm{Phom}_{R-W}(M, N)^K & \xrightarrow{\alpha'} & \mathrm{Phom}_{R^K-W^K}(M^K, N^K) \\
{\scriptstyle \partial_{\mathcal{A}} \otimes 1}\Big\downarrow & & {\scriptstyle \partial_{\mathcal{A}^K}}\Big\downarrow \\
\mathrm{Hom}_R(W_0 \otimes_R M, N)^K & \xrightarrow{\alpha''} & \mathrm{Hom}_{R^K}(W_0^K \otimes_{R^K} M^K, N^K),
\end{array}$$

where α' and α'' are the corresponding natural morphisms. Since M and W are finite-dimensional, the maps α' and α'' are isomorphisms and, therefore, α is an isomorphism too.

Now, we follow Kasjan's argument in [22](2.5). If M^K and N^K have a common direct summand then there exist morphisms $f \in \mathrm{Hom}_{\mathcal{A}^K}(M^K, N^K)$ and $g \in \mathrm{Hom}_{\mathcal{A}^K}(N^K, M^K)$ such that gf is a non-trivial idempotent of M^K. Since α is an isomorphism, there are $f_1, \ldots, f_a \in \mathrm{Hom}_{\mathcal{A}}(M, N)$, $g_1, \ldots, g_b \in \mathrm{Hom}_{\mathcal{A}}(N, M)$, and scalars $\lambda_1, \ldots, \lambda_a, \mu_1, \ldots, \mu_b \in K$, with $f = \alpha[\sum_{i=1}^a f_i \otimes \lambda_i]$ and $g = \alpha[\sum_{j=1}^b g_j \otimes \mu_j]$. The morphism α satisfies $\alpha[h \otimes \lambda] = h^K \lambda$, for $h \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $\lambda \in K$. Thus, it gives an isomorphism of K-algebras $\mathrm{End}_{\mathcal{A}}(M)^K \cong \mathrm{End}_{\mathcal{A}^K}(M^K)$ and $gf = \alpha[\sum_{j=1}^b \sum_{i=1}^a g_j f_i \otimes \mu_j \lambda_i]$. Since $\mathcal{A}$ is a Roiter ditalgebra and M is a finite-dimensional indecomposable, the finite-dimensional algebra $\mathrm{End}_{\mathcal{A}}(M)$ is local; see [8](5.12). But $\sum_{j=1}^b \sum_{i=1}^a g_j f_i \otimes \mu_j \lambda_i$, is a non-zero idempotent in $\mathrm{End}_{\mathcal{A}}(M)^K$ and hence, there exist i_0 and j_0 such that $g_{j_0} f_{i_0}$ is not nilpotent. Then, this composition $g_{j_0} f_{i_0}$ is an isomorphism. So, M is isomorphic to a direct summand of N, because idempotents split in $\mathcal{A}$-Mod. $\square$

Theorem 5.2. *Assume that $\mathcal{B}$ is an initial subalgebra of the admissible ditalgebra $\mathcal{A}$, over the perfect field k. Assume that $\mathcal{A}$ is constructible from the generically tame finite-dimensional basic algebra Λ. Consider the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ and the restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{B}$-modules such that, for any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$ and $M \not\cong E(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{B}$-Mod, the module $R(M)$ is isomorphic in $\mathcal{B}$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*

Proof. Consider the algebraic closure K of the ground field k. Then, we have the commutative squares

$$\begin{array}{ccc} \mathcal{A}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{A}^K\text{-Mod} \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle \overline{R}} \\ \mathcal{B}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{B}^K\text{-Mod} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{A}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{A}^K\text{-Mod} \\ {\scriptstyle E}\big\uparrow & & \big\uparrow{\scriptstyle \overline{E}} \\ \mathcal{B}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{B}^K\text{-Mod}, \end{array}$$

where E and $\overline{E}$ denote extension functors and R and $\overline{R}$ denote restriction functors (see [8](20.2) and [8](20.3)). Fix $d \in \mathbb{N}$. Recall that $M \in \mathcal{A}$-Mod is such that $M \not\cong E(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{B}$-Mod, if and only if $M \not\cong ER(M)$. Since k is perfect, the K-ditalgebra $\mathcal{A}^K$ is almost admissible. From (4.7), we have that $\mathcal{A}^K$ is pregenerically tame.

Then, from (3.5), we know that there is a finite family $\overline{\mathcal{I}}(d)$ of indecomposable $\mathcal{B}^K$-modules such that, for any indecomposable $\mathcal{A}^K$-module $\overline{M}$ with $\dim_K \overline{M} \leq d$ and $\overline{M} \not\cong \overline{E}\,\overline{R}(\overline{M})$ in $\mathcal{A}^K$-Mod, the module $\overline{R}(\overline{M})$ is isomorphic in $\mathcal{B}^K$-Mod to a direct sum of modules in $\overline{\mathcal{I}}(d)$.

Take $M \in \mathcal{A}$-Mod indecomposable with $\dim_k M \leq d$ and such that $M \not\cong ER(M)$ in $\mathcal{A}$-Mod. Consider the decomposition $M^K \cong \overline{M}_1 \oplus \cdots \oplus \overline{M}_t$, a direct sum of indecomposable $\mathcal{A}^K$-modules. Assume, for instance, that $\overline{M}_1 \cong \overline{E}\,\overline{R}(\overline{M}_1)$ in $\mathcal{A}^K$-Mod. Then, we have

$$\begin{aligned} (ER(M))^K &= \overline{E}(R(M)^K) = \overline{E}\,\overline{R}(M^K) \cong \overline{E}\,\overline{R}(\overline{M}_1) \oplus \cdots \oplus \overline{E}\,\overline{R}(\overline{M}_t) \\ &\cong \overline{M}_1 \oplus \overline{E}\,\overline{R}(\overline{M}_2) \oplus \cdots \oplus \overline{E}\,\overline{R}(\overline{M}_t). \end{aligned}$$

Since the indecomposable M and the module $ER(M)$ are such that M^K and $ER(M)^K$ share a direct summand, by (5.1), the module M is a direct summand of $ER(M)$. Say, $ER(M) \cong M \oplus L$. Thus, $R(M) \cong RER(M) \cong R(M) \oplus R(L)$, which implies that $R(L) = 0$, and, hence $L = 0$. Therefore, we get the contradiction $ER(M) \cong M$. Similarly, we have that $\overline{M}_i \not\cong \overline{E}\,\overline{R}(\overline{M}_i)$ in $\mathcal{A}^K$-Mod, for all $i \in [1, t]$. It follows that each $\overline{R}(\overline{M}_i)$ is isomorphic in $\mathcal{B}^K$-Mod to a direct sum of modules of $\overline{\mathcal{I}}(d)$. Hence, this is also the case for $\overline{R}(M^K)$.

If $R(M) \cong N_1 \oplus \cdots \oplus N_s$ is a decomposition of $R(M)$ into indecomposables in $\mathcal{B}$-Mod, then $N_1^K \oplus \cdots \oplus N_s^K = [N_1 \oplus \cdots \oplus N_s]^K \cong R(M)^K = \overline{R}(M^K)$. Thus, each N_i^K is isomorphic to a direct sum of indecomposable modules of $\overline{\mathcal{I}}(d)$. Assume that $\overline{\mathcal{I}}(d) = \{L_1, \ldots, L_m\}$. Consider the class $\mathcal{I}_0(d)$ of indecomposable $\mathcal{B}$-modules which appear as direct summands of $R(N)$, where N runs in the class of indecomposable $\mathcal{A}$-modules with $\dim_k N \leq d$ and $N \not\cong ER(N)$ in $\mathcal{A}$-Mod. Consider also a class of representatives $\mathcal{I}(d)$ of the isomorphism classes in $\mathcal{I}_0(d)$. From (5.1) we get that, for each $i \in [1, m]$, there is, up to isomorphism, at most one $N \in \mathcal{I}_0(d)$ such that L_i is a direct summand of N^K. It follows that $\mathcal{I}(d)$ is a finite family too. Clearly, $R(M)$ is isomorphic to a direct sum of modules of $\mathcal{I}(d)$. $\square$

Remark 5.3. Let $\mathcal{A}$ be an admissible ditalgebra $\mathcal{A}$ and adopt the notation of (3.1). Consider the decomposition $1 = \sum_{i=1}^n e_i$ of the unit of R as a sum of primitive orthogonal central idempotents. Given $M \in \mathcal{A}$-Mod, we can consider its *length* $\ell(M)$, meaning its length as left R-module. Moreover, we can consider its length vector $\underline{\ell}(M) = (\ell(e_1M), \dots, \ell(e_nM))$. Then, under the assumptions of last theorem, we can replace the requirement $\dim_k M \leq d$ by the requirement $\ell(M) \leq d$, and still get a valid statement. Indeed, for a fixed $d \geq 0$, there are only finitely many vectors $\underline{\ell} \in \mathbb{Z}^n$ with non-negative coordinates, such that $\sum_{i=1}^n \ell_i \leq d$; then, for each one of them, we can consider the number $\hat{d}(\underline{\ell}) := \sum_{i=1}^n [D_i : k]\ell_i$ and, then take their maximum $\hat{d} := \max\{\hat{d}(\underline{\ell})\}_{\underline{\ell}}$ and consider the set $\mathcal{I}(\hat{d})$. Then, any $M \in \mathcal{A}$-Mod with $\ell(M) \leq d$, has some length vector $\underline{\ell} = \underline{\ell}(M)$ with $\sum_{i=1}^n \ell_i \leq d$, and satisfies that $\dim_k M = \hat{d}(\underline{\ell}) \leq \hat{d}$, and we can apply the last theorem to this $\hat{d}$.

6. Minimal algebras and principal ideal domains

Definition 6.1. A k-algebra B is called *minimal* iff it is of one of the following two types.

1. $B = T_{D_1 \times D_2}(V)$, where D_1 and D_2 are finite-dimensional division k-algebras and V is a simple D_1–D_2-bimodule.
2. $B = T_D(V)$, where D is a finite-dimensional division k-algebra and V is a simple D–D-bimodule.

Proposition 6.2. *Assume that $B = T_D(V)$ is a pregenerically tame minimal algebra of the second type in (6.1). Then, B is a skew polynomial algebra $D[x, s]$, for some automorphism $s : D \longrightarrow D$. Thus, B is a principal ideal domain and the pregeneric modules coincide with the generic modules. In this case, B has a unique generic module.*

Proof. First, notice that $T_D(V)$ is isomorphic to the skew tensor algebra associated in [16](8.5) to the split exact sequence of D–D-bimodules

$$0 \longrightarrow D \longrightarrow D \oplus V \longrightarrow V \longrightarrow 0.$$

Indeed, if we denote by π the unit element of D, then $D = \pi D$ and $\pi d = d\pi$, for any $d \in D$. Then, the morphism of algebras $\phi : T_D(\pi D \oplus V) \longrightarrow T_D(V)$ defined by the identity morphism $Id : D \longrightarrow D$ and the morphism of D–D-bimodules $\phi : \pi D \oplus V \longrightarrow T_D(V)$ given by $\phi(\pi d + v) = d + v$, induces an isomorphism of algebras $T_D(\pi D \oplus V)/(\pi - 1) \cong T_D(V)$.

Then, if $\dim_D V \geq 2$, from the lemma in [16](8.5), B is strictly wild. Then, from the lemma in [16](8.2), there is a finite field extension K of the ground field k and a B–$K\langle x, y\rangle$-bimodule Z, which is free of finite rank over $K\langle x, y\rangle$ and such that the tensor product functor $Z \otimes_{K\langle x,y\rangle} - :$ $K\langle x, y\rangle$-Mod $\longrightarrow B$-Mod is fully faithful. Now, we proceed as in the proof of (2.9) to construct an infinite family $\{H_p\}_p$ of pairwise non-isomorphic indecomposable pregeneric modules with bounded endolength for the k-algebra $K\langle x, y\rangle$. Then, from [8](31.4), we have the infinite family $\{Z \otimes_{K\langle x,y\rangle} H_p\}_p$ of pairwise non-isomorphic indecomposable pregeneric B-modules with bounded endolength.

It follows that $\dim_D V = 1$. Since D and V are finite-dimensional over k, the left and right dimensions of V over D coincide. Choose any non-zero element $x \in V$. Then, for any

$d \in D$, there is a unique $d^s \in D$ such that $dx = xd^s$. It is easy to see that $d \mapsto d^s$ defines an automorphism of D, and then that $B \cong D[x, s]$.

Assume that a B-module G is pregeneric but not generic, then it is a B-module of finite length. Then, for instance from [21](1.1.22), we know that B is a noetherian algebra and, from [16](7.1), we have that G is a finitely generated B-module. From [11]Section 8.2(2.4), the module G is cyclic, thus $G \cong B$, which has infinite length, or $G \cong B/Bf$, which has finite dimension over k. This means that G has infinite length as a B-module and, therefore, it is generic. The uniqueness of G is a well known fact; see [16](4.7)(6). $\square$

Definition 6.3. Let Γ be an algebra over some ground field k. Assume that Γ is a principal ideal domain (possibly non-commutative). Thus, left and right ideals are principal. Recall that a non-unit element $p \in \Gamma \setminus \{0\}$ is called an *atom* iff it is not a product of two non-invertible elements of Γ (or equivalently $0 \neq \Gamma p$ is a maximal left ideal of Γ). An element $a \in \Gamma$ is called *bounded* iff there is a non-zero two-sided element $b \in \Gamma$ (i.e. such that $\Gamma b = b\Gamma$) with $\Gamma b \subseteq \Gamma a$. The element a is called *centrally bounded* iff the element b described before belongs to the center of Γ. The algebra Γ is *bounded* (resp. *centrally bounded*) iff every $a \in \Gamma$ is bounded (resp. centrally bounded).

Lemma 6.4. *Assume that Γ is a principal ideal domain with noetherian center Z, such that Γ is finitely generated as a Z-module. Then, for any non-invertible element $a \in \Gamma \setminus \{0\}$, there is an element $r \in \Gamma \setminus \{0\}$ such that $ra \in Z$ (or, equivalently, $\Gamma a \cap Z \neq 0$). Thus, the algebra Γ is centrally bounded.*

Proof. Given an atom $p \in \Gamma$, since Γp is a maximal left ideal of Γ, we know that $\mathrm{End}_\Gamma(\Gamma/\Gamma p)$ is a division ring, which has a field K as a center. Assume that $Z \cap \Gamma p = 0$, then there is an injective morphism of rings $\sigma : Z \longrightarrow \mathrm{End}_\Gamma(\Gamma/\Gamma p)$ given by $\sigma(z)[r + \Gamma p] = rz + \Gamma p$, for $z \in Z$ and $r \in \Gamma$. It is clear that σ restricts to an injective morphism on the center K of $\mathrm{End}_\Gamma(\Gamma/\Gamma p)$. Hence, the field of fractions Q of the domain Z embeds in K. On the other hand, $\mathrm{End}_\Gamma(\Gamma/\Gamma p)$ is a finitely generated Z-module, because Γ and $\Gamma/\Gamma p$ are finitely generated Z-modules (apply the same argument used to prove [3]II(1.1)(a)). Thus, from the noetherianity of Z, the Z-submodule Q of $\mathrm{End}_\Gamma(\Gamma/\Gamma p)$ should be finitely generated too. But it is well known that it is not the case.

Then, $Z \cap \Gamma p \neq 0$. If $0 \neq z \in Z \cap \Gamma p$, we have $rp = z \in Z$, for some $r \in \Gamma$, and z is a two-sided element of Γ with $\Gamma z \subseteq \Gamma p$. Thus, every atom p of Γ is bounded.

Now, we proceed to prove that every non-invertible element a of Γ is bounded, by induction on the number n of atoms in the atomic decomposition $a = p_1 \cdots p_n$. Assume that $n > 1$, make $a' := p_1 \cdots p_{n-1}$ and suppose that we already know the existence of an element $r' \in \Gamma$ such that $r'a' = z' \in Z$. But we also know there is an $r \in \Gamma$ such that $rp_n = z \in Z$. Then, $rr'a = rr'a'p_n = rz'p_n = rp_nz' = zz' \in Z$. Hence, $\Gamma zz' \subseteq \Gamma a$, and we are done. $\square$

It is convenient to have an explicit description for the almost split sequences, up to isomorphism, in the category of finite length modules Γ-mod over a bounded principal ideal domain Γ. This result is presumably known, but we outline a proof for the convenience of the reader. Recall that the non-zero elements $a, b \in \Gamma$ are called *similar* iff $\Gamma/\Gamma a \cong \Gamma/\Gamma b$, and that similarity is an equivalence relation in $\Gamma \setminus \{0\}$.

Lemma 6.5. *Let Γ be a bounded principal ideal domain. For each atom $p \in \Gamma$, consider the corresponding simple Γ-module $S_p := \Gamma/\Gamma p$. Then, for each $i \in \mathbb{N}$, up to isomorphism, there is* *a unique indecomposable Γ-module E_i^p with length i and all composition factors isomorphic to S_p. The family $\{E_i^p \mid i \in \mathbb{N}, p \in P\}$, where P denotes a set of representatives of the similarity classes of all atoms of Γ, is a complete set of representatives of the isoclasses of the indecomposable Γ-modules of finite length. Moreover, there are almost split sequences:*

$$\zeta_1^p : E_1^p \longrightarrow E_2^p \longrightarrow E_1^p,$$
$$\zeta_n^p : E_n^p \longrightarrow E_{n+1}^p \oplus E_{n-1}^p \longrightarrow E_n^p, \quad \textit{for } n \geq 2.$$

Proof. Observe that $\Gamma b/\Gamma ab \cong \Gamma/\Gamma a$, for all non-zero $a, b \in \Gamma$. Then, whenever we have a product $c = p_1 p_2 \cdots p_t$ of atoms in Γ, which are similar to the atom $p \in \Gamma$, we have the composition series

$$0 = \frac{\Gamma p_1 p_2 \cdots p_t}{\Gamma p_1 p_2 \cdots p_t} \subseteq \frac{\Gamma p_2 \cdots p_t}{\Gamma p_1 p_2 \cdots p_t} \subseteq \cdots \subseteq \frac{\Gamma p_t}{\Gamma p_1 p_2 \cdots p_t} \subseteq \frac{\Gamma}{\Gamma p_1 p_2 \cdots p_t} = \Gamma/\Gamma c$$

with composition factors $[\frac{\Gamma p_i \cdots p_t}{\Gamma p_1 \cdots p_t}]/[\frac{\Gamma p_{i+1} \cdots p_t}{\Gamma p_1 \cdots p_t}] \cong \frac{\Gamma p_i \cdots p_t}{\Gamma p_{i+1} \cdots p_t} \cong \frac{\Gamma}{\Gamma p_{i+1}} \cong S_p$. Therefore, $\Gamma/\Gamma c$ is a Γ-module with length t and all composition factors isomorphic to S_p.

From [11]Section 8.2(2.4), every finitely generated Γ-module is a direct sum of cyclic modules. Since Γ has infinite length, we concentrate on torsion cyclic modules. From [12](1.5.6), since Γ is a bounded principal ideal domain, every torsion cyclic indecomposable Γ-module has the form $\Gamma/\Gamma p_1 \cdots p_t$, where $p_1, \ldots, p_t$ are similar atoms of Γ. From [20](3.21), a cyclic Γ-module $\Gamma/\Gamma p_1 p_2 ... p_t$, where $p_1, p_2, \ldots, p_t$ are atoms similar to a fixed atom p of Γ, is indecomposable iff the annihilator of $\Gamma/\Gamma p_1 p_2 \cdots p_t$ is Γq^t, where Γq is the annihilator of $\Gamma/\Gamma p$. Now, assume that $\Gamma/\Gamma p_1 \cdots p_t$ and $\Gamma/\Gamma q_1 \cdots q_t$ are indecomposable Γ-modules where all the atoms $p_1, \ldots, p_t, q_1, \ldots, q_t$ are similar, say to the atom p. Since Γ is bounded, from [12](1.5.6), we know they are isomorphic iff they have the same annihilator, which is the case because they both have annihilator Γq^t. We have seen that for each $p \in P$ and $i \in \mathbb{N}$, there is at most one indecomposable Γ-module (up to isomorphism) with length i and composition factors isomorphic to S_p.

Fix an atom $p \in \Gamma$, consider the annihilator Γq of $\Gamma/\Gamma p$ and, for each $t \in \mathbb{N}$, consider the algebra $\Gamma_t := \Gamma/\Gamma q^t$. Thus, Γ_t is a left artin ring and the ideal $q\Gamma_t = \Gamma q/\Gamma q^t$ is nilpotent with $\Gamma_t/q\Gamma_t \cong \Gamma/\Gamma q$ semisimple. Thus, $\operatorname{rad} \Gamma_t = q\Gamma_t$ and the algebra Γ_t has only one isomorphism class of simples, namely that of S_p. It follows that there is only one isomorphism class of indecomposable projective Γ_t-modules, choose a representative P_t of this class. Denote by m the multiplicity of the simple module in the semisimple decomposition of $\Gamma/\Gamma q$, thus $\Gamma/\Gamma q \cong mS_p$. Since $\Gamma_t/q\Gamma_t \cong \Gamma/\Gamma q \cong mS_p$ and, hence, $\Gamma_t \cong mP_t$, we have $\Gamma_{t-1} \cong \Gamma q/\Gamma q^t = \operatorname{rad} \Gamma_t \cong \operatorname{rad}(mP_t) = m(\operatorname{rad} P_t)$. It follows that $\operatorname{rad} P_t$ is an indecomposable projective Γ_{t-1}-module, thus $P_{t-1} \cong \operatorname{rad} P_t = (q\Gamma_t)P_t = qP_t$. Then, we have the composition series of P_t,

$$\{0\} = q^t P_t \subseteq q^{t-1} P_t \subseteq \cdots \subseteq q^2 P_t \subseteq qP_t \subseteq P_t,$$

with all composition factors $q^i P_t/q^{i+1} P_t \cong S_p$ and every term in the filtration is an indecomposable Γ-module with local endomorphism algebra. Having in mind the previous paragraph, now we know that, for each $p \in P$ and each $t \in \mathbb{N}$, up to isomorphism, there is exactly one indecomposable Γ-module E_t^p with length t and composition factors isomorphic to S_p (with the previous notation, $E_t^p \cong P_t$).

Notice that $\operatorname{Hom}_\Gamma(E_{t_1}^{p_1}, E_{t_2}^{p_2}) = 0$, whenever p_1 and p_2 are not similar atoms. Indeed, consider the annihilator Γq_1 of $\Gamma/\Gamma p_1$ and the annihilator Γq_2 of $\Gamma/\Gamma p_2$; then, if there is a non-zero morphism of Γ-modules $f : E_{t_1}^{p_1} \longrightarrow E_{t_2}^{p_2}$, then the submodule $\operatorname{Im} f \subseteq E_{t_2}^{p_2}$ has a non-trivial annihilator Γa such that $\Gamma q_1^{t_1} = \operatorname{Ann}(E_{t_1}^{p_1}) \subseteq \Gamma a \supseteq \operatorname{Ann}(E_{t_2}^{p_2}) = \Gamma q_2^{t_2}$. It follows that the two-sided elements q_1 and q_2 share an atom of their atom decompositions. Thus, p_1 and p_2 are similar.

Fix $p \in P$, let Γq be the annihilator of $\Gamma/\Gamma p$, and $\Gamma_t = \Gamma/\Gamma q^t$, for $t \geq 2$. Let us describe the almost split sequences in Γ_t-mod. Again, if P_t denotes a fixed representative of the unique isoclass of projective indecomposable Γ_t-modules, the modules $q^{t-1}P_t, \dots, qP_t, P_t$ represent all the non-isomorphic indecomposable Γ_t-modules. Moreover, the modules $q^{t-1}P_t, \dots, qP_t$ are not injective.

We look first at the case $t = 2$. We only have two non-isomorphic indecomposable Γ_2-modules qP_2 and P_2. Consider the almost split sequence starting at qP_2 in Γ_2-mod

$$\chi_1[2] : 0 \longrightarrow qP_2 \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Here, N is indecomposable and $N \ncong P_2$, because P_2 is a projective Γ_2-module. Thus, $N \cong qP_2$ and the length of M is 2. Since the given sequence does not split, $M \cong P_2$, and we have an almost split sequence in Γ_2-mod of the form

$$\xi_1^p[2] : 0 \longrightarrow qP_2 \longrightarrow P_2 \longrightarrow qP_2 \longrightarrow 0.$$

We claim that, for $t \geq 3$, the category Γ_t-mod has almost split sequences of the form:

$$\xi_{t-1}^p[t] : 0 \longrightarrow q^{t-1}P_t \longrightarrow q^{t-2}P_t \longrightarrow q^{t-1}P_t \longrightarrow 0$$

and, for $i \in [1, t-2]$,

$$\xi_i^p[t] : 0 \longrightarrow q^iP_t \longrightarrow q^{i-1}P_t \oplus q^{i+1}P_t \longrightarrow q^iP_t \longrightarrow 0.$$

We proceed by induction on $t \geq 3$. Thus, by induction hypothesis, we have almost split sequences $\xi_1^p[t-1], \dots, \xi_{t-2}^p[t-1]$ in Γ_{t-1}-mod ending at $qP_{t-1}, \dots, q^{t-2}P_{t-1}$, respectively. As we pointed out before, $q^{t-1}P_t, \dots, qP_t, P_t$ represent all the non-isomorphic indecomposable Γ_t-modules, all of them are Γ_{t-1}-modules with the only exception of P_t. For $i \in [1, t-1]$, consider an almost split sequence in Γ_t-mod starting at q^iP_t

$$\chi_i[t] : 0 \longrightarrow q^iP_t \longrightarrow M_i \longrightarrow N_i \longrightarrow 0.$$

Again, the indecomposable Γ_t-module N_i is not projective, hence $N_i \ncong P_t$. Thus, $N_i \in \Gamma_{t-1}$-mod. Consider first the case $i > 1$. Here, if P_t was a direct summand of M_i, there would be an irreducible morphism $q^iP_t \longrightarrow P_t$, which is impossible because the inclusion $qP_t \longrightarrow P_t$ is the unique irreducible map in Γ_t-mod ending at P_t. Thus, P_t is not a direct summand of M_i and $M_i \in \Gamma_{t-1}$-mod. Thus, the whole sequence $\chi_i[t]$ lies in Γ_{t-1}-mod and, having in mind that $q^iP_t \cong q^{i-1}P_{t-1}$, from the uniqueness of almost split sequences in Γ_{t-1}-mod, we obtain that $\chi_i[t] \cong \xi_{i-1}^p[t-1]$ in Γ_{t-1}-mod. Then, there is an almost split sequence $\xi_i^p[t]$ in Γ_t-mod of the desired form (in fact, $\xi_i^p[t] \cong \xi_{i-1}^p[t-1]$ in Γ_{t-1}-mod). Now, we consider the remaining case $i = 1$. Recall that we have already chosen an almost split sequence $\chi_1[t]$ in Γ_t-mod starting at qP_t. Again, it is clear that $N_1 \ncong P_t$. Thus, N_1 must be isomorphic to one of the Γ_t-modules $q^{t-1}P_t, \dots, q^2P_t, qP_t$. If $N_1 \cong q^jP_t$, with $j > 1$, then we have the almost split sequence $\xi_j^p[t]$ in Γ_t-mod ending and starting at q^jP_t, contradicting the fact that $qP_t \ncong q^jP_t$. Thus, $N_1 \cong qP_t$. Moreover, the existence of the irreducible morphism $qP_t \longrightarrow P_t$ in Γ_t-mod implies that $M_1 \cong P_t \oplus Q_t$, for some $Q_t \in \Gamma_t$-mod. Consider an indecomposable direct summand q^jP_t of Q_t, say $Q_t \cong q^jP_t \oplus C_t$, for some $C_t \in \Gamma_t$-mod. Assume that $j > 2$ (notice that, by length considerations on the exact sequence $\chi_1[t]$, this is the case if $C_t \neq 0$). Then, there is an irreducible morphism $q^j P_t \longrightarrow qP_t$ in Γ_t-mod, thus there is an irreducible morphism $q^{j-1}P_{t-1} \longrightarrow P_{t-1}$ in Γ_{t-1}-mod, which is impossible. Then, $j \leq 2$ and $C_t = 0$. Again, counting lengths, we see that $j = 2$. Then, there is an almost split sequence $\xi_1^p[t]$ of the desired form, in fact $\xi_1^p[t] \cong \chi_1[t]$, and we have described all the almost split sequences in Γ_t-mod.

We have that given a non-projective indecomposable E_n^p of Γ-mod and $t \in \mathbb{N}$ such that q^t annihilates E_{n+1}^p, there is an almost split sequence in Γ_t-mod

$$\zeta_n^p[t] : E_n^p \longrightarrow E_{n+1}^p \oplus E_{n-1}^p \longrightarrow E_n^p$$

of the form specified in the statement of the lemma, here $\zeta_n^p[t] \cong \xi_{t-n}^p[t]$. It depends on t. For each indecomposable E_n^p, consider the almost split sequence $\zeta_n^p := \zeta_n^p[n+1]$. We claim that $\zeta_1^p, \zeta_2^p, \ldots, \zeta_n^p, \ldots$ are almost split sequences in Γ-mod. Fix $n \in \mathbb{N}$ and let us see that $\zeta_n^p : E_n^p \xrightarrow{\sigma} M \xrightarrow{\pi} E_n^p$ is almost split in Γ-mod. Take a non-retraction morphism $h : N \longrightarrow E_n^p$ in Γ-mod with N indecomposable. We want to show that h factors through π. This is clear if $N \cong E_m^{p'}$ with p' not similar to p, because, as we saw before, h must be zero. It remains the case $N \cong E_m^p$, for some $m \in \mathbb{N}$. Then, for $t > n+1$ big enough, we have an almost split sequence $\zeta_n^p[t]$ ending at E_n^p in Γ_t-mod with $E_m^p \in \Gamma_t$-mod. Then, ζ_n^p and $\zeta_n^p[t]$ are both almost split sequences in Γ_{n+1}-mod ending at E_n^p; hence there is an isomorphism of sequences $\zeta_n^p \cong \zeta_n^p[t]$ in Γ-mod. Thus, the morphism h factors through π in Γ-mod, and we are done. $\square$

Remark 6.6. The description of generically tame minimal algebras of type 2 is given in (6.2). The description of generically tame minimal algebras of type 1, which are of infinite representation type, is the following:

1. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 2 = \dim M_F$; or
2. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 4$ and $\dim M_F = 1$, or $\dim_G M = 1$ and $\dim M_F = 4$.

This follows from the work of Dlab and Ringel in [17] and the fact that tame hereditary algebras coincide with generically tame hereditary algebras of infinite representation type (see [16](8.4)). Here, following Crawley-Boevey, we say that a connected finite-dimensional hereditary algebra is *tame hereditary* if its quadratic form is positive semidefinite but not positive definite; see [16](8.3). Then, Theorem 1.1 will follow from (8.2).

Recall that given an algebra B, as in 1 or 2, the quadratic form of B is semidefinite positive and its radical is generated by a vector $\underline{\lambda} \in \mathbb{Z}^2$ with positive components. A *regular* B-module is a finite-dimensional B-module whose indecomposable direct summands H have length vector of the form $\underline{\ell}(H) = c_H \underline{\lambda}$, for some $c_H \in \mathbb{N}$; see [26](4.1). The full subcategory B-reg of B-mod formed by the regular modules is abelian and uniserial; see [26](4.2). The simple objects in B-reg are called *simple regular* modules. The class of indecomposable finite-dimensional B-modules is the disjoint union of three classes: the preprojective ones, the preinjective ones and the regular ones. The non-regular indecomposable B-modules satisfy that, for any $c \in \mathbb{N}$, they admit only finitely many isoclasses with length bounded by c.

Given any admissible ditalgebra $\mathcal{A}$, we say that an $\mathcal{A}$-module M is *homogeneous* iff there is an almost split conflation in $\mathcal{A}$-mod of the form $M \to L \to M$. It is known that in the previous context, every indecomposable regular B-module is homogeneous in B-mod.

Lemma 6.7. *Assume that Γ is a bounded principal ideal domain and that B is a finite-dimensional generically tame hereditary k-algebra of infinite representation type. Assume the existence of a finite-dimensional algebra B' and an epimorphism of k-algebras $\psi : B' \longrightarrow B'_\Sigma$, which is a universal localization, as in* [15] *Section* 2*, where B' is Morita equivalent to B and B'_Σ is Morita equivalent to Γ. Then, we have the following.*

1. *Every simple Γ-module is finite-dimensional.*
2. *Consider the composition H of the functors*

$$\Gamma\text{-Mod} \xrightarrow{G_1} B'_\Sigma\text{-Mod} \xrightarrow{F_\psi} B'\text{-Mod} \xrightarrow{G_2} B\text{-Mod},$$

*where G_1 and G_2 are equivalence functors, and F_ψ is the restriction functor. Then, the almost split sequence ending at any $M \in \Gamma$-*mod *is mapped by H onto an almost split sequence in B-*mod*, whenever $H(M)$ is a homogeneous B-module.*

Proof. (1) Fix an atom $p \in \Gamma$. Since Γ is bounded, there is a two-sided element $p^* \in \Gamma$ such that $\Gamma p^* \subseteq \Gamma p$ and Γp^* is the unique maximal two-sided ideal with this property. Then, from theorem [12](1.5.4), the element p^* is a two-sided atom, we have that $\mathrm{Ann}_\Gamma(\Gamma/\Gamma p) = \Gamma p^*$ and $\Gamma/\Gamma p^*$ is a simple artinian ring. From [16](4.7)(2), the simple Γ-module $\Gamma/\Gamma p$ has finite endolength. Assume now that $\Gamma/\Gamma p$ is infinite-dimensional over k, thus $\Gamma/\Gamma p$ is a pregeneric Γ-module. The equivalences G_1 and G_2 have the form $G_1 \cong P_1 \otimes_\Gamma -$ and $G_2 \cong P_2 \otimes_{B'} -$, for some bimodules P_1 and P_2, where P_1 is finitely generated projective right Γ-module and P_2 is finitely generated projective right B'-module.

Since P_1 is a finitely generated projective right Γ-module, and Γ is a principal ideal domain, it is a free right Γ-module of finite rank. Then, from [8](31.4), $G_1(\Gamma/\Gamma p)$ is a pregeneric B'_Σ-module. From (2.2), we have that $F_\psi G_1(\Gamma/\Gamma p)$ is a pregeneric B'-module. From [8](29.8), we know that $G_2 F_\psi G_1(\Gamma/\Gamma p)$ is a pregeneric B-module and, hence, a generic B-module. On the other hand, from [16](4.7)(6), the generic Γ-module is the restriction of the simple module Q of the simple artinian quotient ring of Γ. As before, $G_2 F_\psi G_1(Q)$ is a generic B-module. Now, as we reminded in (6.6), Crawley-Boevey has shown that B is generically tame of infinite representation type iff it is tame hereditary. Ringel has shown in [27]Section 6 that, in this case, the algebra B admits a unique generic module. Thus, $G_2 F_\psi G_1(Q) \cong G_2 F_\psi G_1(\Gamma/\Gamma p)$ and, therefore, $Q \cong \Gamma/\Gamma p$. This is not possible because $\mathrm{Ann}_\Gamma(Q) = 0 \neq \mathrm{Ann}_\Gamma(\Gamma/\Gamma p)$.

(2) We already know that G_1 and G_2 preserve almost split sequences. Since $\psi : B' \longrightarrow B'_\Sigma$ is a universal localization, Σ is a set of morphisms between finitely generated left B'-modules and the restriction functor $F_\psi : B'_\Sigma\text{-Mod} \longrightarrow B'\text{-Mod}$ identifies B'_Σ-Mod with the full subcategory of B'-Mod determined by the B'-modules M with $\mathrm{Hom}_{B'}(\sigma, M)$ isomorphism, for all $\sigma \in \Sigma$; see [24]Section 6. Thus, B'_Σ-Mod is identified with an extension closed full subcategory of B'-Mod. Assume that $M \in \Gamma$-mod is indecomposable with $H(M)$ homogeneous. Consider an almost split sequence ξ in Γ-mod ending at M and make $N := G_1(M)$. Then, $G_1(\xi) \in \mathrm{Ext}_{B'_\Sigma}(N, N) \cong \mathrm{Ext}_{B'}(F_\psi N, F_\psi N)$. But since $H(M)$ is homogeneous, so is $F_\psi N$. Since F_ψ is a full and faithful functor, $F_\psi G_1(\xi)$ generates the socle of the $\mathrm{End}_{B'}(F_\psi N)$-module $\mathrm{Ext}_{B'}(F_\psi N, F_\psi N)$. It follows that $F_\psi G_1(\xi)$ is an almost split sequence, and, hence, so is $H(\xi)$. $\square$

Theorem 6.8. *For any generically tame minimal algebra B of infinite representation type, there is a bounded principal ideal domain Γ and an exact full and faithful functor $H : \Gamma\text{-Mod} \longrightarrow B\text{-Mod}$ such that*

1. *every simple Γ-module is finite-dimensional;*
2. *H maps almost split sequences of Γ-mod onto almost split sequences in B-mod;*
3. *for each $d \in \mathbb{N}$ and almost every $M \in B$-mod with $\dim_k M \leq d$, there is $N \in \Gamma$-mod with $H(N) \cong M$.*

Proof. Assume that B is a generically tame minimal algebra of infinite representation type.

Suppose first that B is of the second type in (6.1). From (6.2), we can assume that $B = D[x, s]$. It is well known that $\Gamma := D[x, s]$ is a bounded principal ideal domain (see [21](3.15)) and, since Γ satisfies the Euclidean algorithm, the simple modules are finite-dimensional. Then, taking as H the identity functor, we see that 2 and 3 are clear in this case.

Now, assume that B is of the first type in (6.1). Then, as remarked in (6.6), we are in one of the following cases:

1. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 2 = \dim M_F$; or
2. B is the matrix algebra $\begin{pmatrix} F & 0 \\ M & G \end{pmatrix}$, where F and G are finite-dimensional division k-algebras and M is a simple G–F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 4$ and $\dim M_F = 1$, or $\dim_G M = 1$ and $\dim M_F = 4$.

For case 1, Dlab and Ringel constructed in [18] a bounded principal ideal domain Γ and a full and faithful functor $H : \Gamma\text{-Mod} \longrightarrow B\text{-Mod}$, which induces an equivalence between the category Γ-mod, of the finite-dimensional Γ-modules, onto a full subcategory of the category B-reg, of the finite-dimensional regular B-modules, such that $B\text{-reg} = \operatorname{Im} H \coprod \mathcal{U}$, where $\mathcal{U}$ is a uniserial subcategory of B-reg with global dimension one and with only one simple object E_1. In fact, $\mathcal{U} = \mathcal{U}(E_1)$ as defined in [26](1.2), where E_1 is a simple regular B-module. Moreover, H can be realized as the composition of the usual Morita equivalence functor $\Gamma\text{-Mod} \longrightarrow M_2(\Gamma)\text{-Mod}$ with the restriction functor $F_\psi : M_2(\Gamma)\text{-Mod} \longrightarrow B\text{-Mod}$ induced by an algebra epimorphism $\psi : B \longrightarrow M_2(\Gamma)$. Moreover, the epimorphism ψ is a universal localization; see [15](5.3). Then, from (6.7), the simple Γ-modules are finite-dimensional and H preserves almost split sequences (indeed every indecomposable regular B-module is homogeneous; see [17]).

Moreover, $\operatorname{Ext}_B(E_1, E_1)$ is one-dimensional as a left $\operatorname{End}_B(E_1)$-space, since the almost split sequence in B-mod ending at E_1 starts at E_1 and generates the socle of the $\operatorname{End}_B(E_1)$-module $\operatorname{Ext}_B(E_1, E_1)$. Then, we can follow Ringel's argument in [28](3)(3.1)(2), to show the existence of almost split sequences in $\mathcal{U}$ of the form

$$\zeta_1 : E_1 \longrightarrow E_2 \longrightarrow E_1,$$
$$\zeta_n : E_n \longrightarrow E_{n+1} \oplus E_{n-1} \longrightarrow E_n, \quad \text{for } n \geq 2,$$

and the fact that any indecomposable $N \in \mathcal{U}$ is isomorphic to some E_n. Indeed, the first sequence ζ_1 is the generator of the one-dimensional $\operatorname{End}_B(E_1)$-space $\operatorname{Ext}_B(E_1, E_1)$, and for the inductive step of the construction one uses that $\operatorname{Ext}^2_B(E_1, E_1) = 0$. Then, it follows that $\mathcal{U}$ admits only finitely many isoclasses of modules of any given dimension, and item 3 follows.

For case 2, Crawley-Boevey constructed in [15](5.3) a principal ideal domain Γ, a finite-dimensional algebra B' and an epimorphism of k-algebras $\psi : B' \longrightarrow B'_\Sigma$, which is a universal localization, where B' is Morita equivalent to B and B'_Σ is Morita equivalent to Γ. Moreover, the ring Γ is finitely generated over its center Z, which is a Dedekind domain. From (6.4), we know that Γ is a bounded principal ideal domain. Consider now the composition functor H defined in (6.7)

$$\Gamma\text{-Mod} \xrightarrow{G_1} B'_\Sigma\text{-Mod} \xrightarrow{F_\psi} B'\text{-Mod} \xrightarrow{G_2} B\text{-Mod},$$

and notice that we already know that the simple Γ-modules are finite-dimensional.

We have the exact full and faithful functor $H : \Gamma\text{-mod} \longrightarrow B\text{-mod}$ which induces an isomorphism $\mathrm{Ext}_\Gamma(M, M) \cong \mathrm{Ext}_B(H(M), H(M))$, for each finite-dimensional indecomposable Γ-module M. From Lemma 6.5, we obtain that $\mathrm{Ext}_B(H(M), H(M)) \neq 0$, thus $H(M)$ cannot be preprojective or preinjective, see [3]VIII.1.1.7. Thus, $H(M)$ is a regular B-module. But every indecomposable regular B-module is homogeneous. Then, from (6.7), the functor H preserves almost split sequences.

It follows from Crawley-Boevey's arguments in [15], that in this case we also get $B\text{-reg} \cong \mathrm{Im}\, H \coprod \mathcal{U}$, with $\mathcal{U}$ uniserial with global dimension one and generated by a simple regular module and, then, proceeding as before, we can complete the proof of item 3. In the following paragraphs, we detail the arguments.

The universal localization $\psi : B' \longrightarrow B'_\Sigma$ that we are considering is defined by a simple regular B'-module S', and we have the full subcategories $\mathcal{M}_r = \{M \in B'\text{-mod} \mid \mathrm{Hom}_{B'}(S', M) = 0 \text{ and } \mathrm{Ext}_{B'}(S', M) = 0\}$ and $\mathcal{M}_t = \{M \in B'\text{-mod with no projective direct summands} \mid \mathrm{Hom}_{B'}(M, S') = 0 \text{ and } \mathrm{Hom}_{B'}(S', M) = 0\}$. The functor F_ψ determines an equivalence $B'_\Sigma\text{-mod} \longrightarrow \mathcal{M}_r$, while the functor $B'_\Sigma \otimes_{B'} -$ determines an equivalence $\mathcal{M}_t \longrightarrow B'_\Sigma\text{-mod}$. Indeed, the first equivalence follows from [15](2.5). For the second one, recall from [15](2.3) that the functor $B'_\Sigma \otimes_{B'} -$ determines an equivalence $\mathcal{M}_t \longrightarrow \mathcal{M}'$, where $\mathcal{M}'$ is the category of finitely presented B'_Σ-modules with no projective direct summand, and we claim that $\mathcal{M}' = B'_\Sigma\text{-mod}$. This is so because the bounded principal ideal domain Γ is Morita equivalent to B'_Σ. So, the algebra B'_Σ is noetherian because Γ is so and, hence, the finitely presented B'_Σ-modules coincide with the finitely generated ones. By Morita's Theorem, $G_1 \cong Q \otimes_\Gamma -$, for some B'_Σ–Γ-bimodule Q which is finitely generated projective (hence free of finite rank) by the right. Thus, G_1 maps finite-dimensional Γ-modules onto finite dimensional B'_Σ-modules. Moreover, $G_1(\Gamma) \cong Q$ is the unique, up to isomorphism, indecomposable projective B'_Σ-module and it has infinite dimension. It follows that $\mathcal{M}' = B'_\Sigma\text{-mod}$, as claimed.

From the Auslander–Reiten formula, $D\mathrm{Ext}_{B'}(M, N) \cong \underline{\mathrm{Hom}}_{B'}(\tau^{-1}N, M)$, where $\tau^{-1} = TrD$ is the Auslander–Reiten translation of B'-mod, see [2], we obtain that $\mathcal{M} := \mathcal{M}_r \cap B'\text{-reg} = \mathcal{M}_t \cap B'\text{-reg}$.

As remarked in [15](2.4), the functor $B'_\Sigma \otimes_{B'} -$ induces a bijection from the set of isomorphism classes of ∂-simples in B'-mod different from S' to the set of isomorphism classes of σ-simples in B'_Σ-mod. Here ∂ is the normalized defect for B' and σ is the rank function on B'_Σ induced by ∂; see [15](2.2) and [15](4.1). Also, from [15](4.2)(4), we have that B'_Σ is a maximal order. Moreover, from [15](3.1)(2 and the paragraph before), the σ-simples are the simple B'_Σ-modules and, from [15](4.1), the simple regular B'-modules are the ∂-simple B'-modules. Thus, $\mathcal{M}$ contains all the simple regular B'-modules different from S' and $S' \notin \mathcal{M}$.

Now, F_ψ maps simple B'_Σ-modules onto simple regular B'-modules, and every simple regular B'-module in $\mathcal{M}$ has the form $F_\psi(T)$, for some simple B'_Σ-module T. Indeed, the simple Γ-modules are homogeneous and admit an almost split sequence with indecomposable middle term. Then, the same is true for B'_Σ. Then, if T is a simple B'_Σ-module, the functor F_ψ induces an isomorphism $\mathrm{Ext}_{B'_\Sigma}(T,T) \cong \mathrm{Ext}_{B'}(F_\psi T, F_\psi T)$ which maps the simple socle, generated by an almost split sequence ξ, of the $\mathrm{End}_{B'_\Sigma}(T)$-module at the left on the simple socle, generated by the almost split sequence $F_\psi(\xi)$, of the $\mathrm{End}_{B'}(F_\psi T)$-module at the right. Recall that we already know that $F_\psi T$ is regular, hence homogeneous. But, clearly the almost split sequence $F_\psi(\xi)$ has an indecomposable middle term, and hence $F_\psi(T)$ is a simple regular B'-module. With the same argument, we see that if S is a simple regular B'-module in $\mathcal{M}$, say of the form $F_\psi T'$, with $T' \in B'_\Sigma$-mod, then T' is a simple B'_Σ-module.

Then, H maps simple Γ-modules onto simple regular B-modules and every regular simple B-module has the form $H(T)$, for some simple Γ-module T, with the only exception of $S := G_2(S')$. Since H preserves almost split sequences, we obtain that B-reg $= \mathrm{Im}\, H \coprod \mathcal{U}$, where $\mathcal{U} = \mathcal{U}(S)$ is the uniserial subcategory of B-reg generated by S, as described above (see [26]). $\square$

7. Reduction to minimal algebras

The proof of the next theorem requires an induction argument which uses the following norm (see [4]).

Definition 7.1. Assume that $\mathcal{A}$ is an admissible k-ditalgebra with layer (R,W), as in (3.1). Then, for $M \in \mathcal{A}$-Mod, we define its *norm* as the number

$$\|M\| := \dim_k \mathrm{Hom}_R(W_0 \otimes_R M, M).$$

If $1 = \sum_{i=1}^n e_i$ is the decomposition of the unit of R as a sum of orthogonal primitive central idempotents, the length vector of M is given by

$$\underline{\ell}(M) = (\ell_{D_1}(e_1 M), \ldots, \ell_{D_n}(e_n M)),$$

and the length of M is $\ell(M) := \ell_R(M) = \sum_{i=1}^n \ell_{D_i}(e_i M)$. The *support* of M is the set of idempotents e_i with $e_i M \neq 0$. The $\mathcal{A}$-module M is called *sincere* iff $e_i M \neq 0$, for all $i \in [1,n]$.

For $M \in \mathcal{A}$-Mod, we have $\|M\| = \sum_{i,j} \ell_{D_i}(e_i M)\ell_{D_j}(e_j M)\dim_k(e_i W_0 e_j)$. Consequently, for $\underline{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n$, with non-negative entries, its norm is defined by

$$\|\underline{\ell}\| = \sum_{i,j} \ell_i \ell_j \dim_k(e_i W_0 e_j).$$

Lemma 7.2. *Let $\mathcal{A}$ be an admissible ditalgebra with layer (R,W), as in (3.1). Assume that $W_0 \neq 0$ and that M is a sincere indecomposable $\mathcal{A}$-module with $\|M\| \leq d$, then the length of M satisfies $\ell(M) \leq nd$.*

Proof. Consider the decomposition $1 = \sum_{j=1}^n e_j$ of the unit of R as a sum of primitive central orthogonal idempotents. Consider the length vector $\underline{\ell} = \underline{\ell}(M)$ of a sincere $\mathcal{A}$-module M with $\|M\| \leq d$. Then $\sum_{i,j} \ell_i \ell_j \dim_k(e_i W_0 e_j) = \|M\| \leq d$. The statement is clear for $n = 1$, so assume that $n \geq 2$. Given $i \in [1,n]$, we have

$$\ell_i \left[\sum_j \ell_j \dim_k(e_i W_0 e_j)\right] \leq d \quad \text{and} \quad \ell_i \left[\sum_j \ell_j \dim_k(e_j W_0 e_i)\right] \leq d.$$

If $\sum_j \ell_j \dim_k(e_i W_0 e_j) = 0$ and $\sum_j \ell_j \dim_k(e_j W_0 e_i) = 0$, then $e_i W_0 e_j = 0 = e_j W_0 e_i$, for all $j \in [1,n]$. If we make $e = \sum_{j \neq i} e_j$, then $W_0 = e W_0 e$ and A splits as a product of algebras

$A \cong D_i \times T_{Re}(W_0)$. Then, there is no sincere indecomposable A-module. Thus, if M is a sincere indecomposable $\mathcal{A}$-module then, for each $i \in [1, n]$, there is a positive integer c_i with $\ell_i c_i \leq d$. Thus, $\ell_i \leq d$, for all i, and $\ell(M) \leq nd$, as claimed. □

Lemma 7.3. *Let k be a perfect field and let $\mathcal{A}$ be an admissible ditalgebra with layer (R, W). Assume that $\mathcal{A}^X$ is obtained from $\mathcal{A}$ by reduction, using the $\mathcal{B}$-module X, where $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$ and X is a finite direct sum of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{B}$-modules. Then, the algebra $\mathrm{End}_B(X)^{op}$ admits the splitting $\mathrm{End}_B(X)^{op} = S \oplus P$, where P is the radical, and $\mathcal{A}^X$ is an admissible ditalgebra with triangular layer (S, W^X). Let $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ be the associated functor. Suppose that the subalgebra $\mathcal{B}$ of $\mathcal{A}$ is determined by the R–R-bimodule decomposition $W_0 = W_0' \oplus W_0''$. Then, we have the following.*

1. *We have $\|F^X(N)\| - \|N\| = \dim_k \mathrm{Hom}_R(W_0' \otimes_R F^X(N), F^X(N))$, for any $N \in \mathcal{A}^X\text{-Mod}$. Thus, $\|N\| < \|F^X(N)\|$, whenever $F^X(N)$ is a sincere $\mathcal{A}$-module and $W_0' \neq 0$.*
2. *If we denote by $R^{\mathcal{A}}_{\mathcal{B}} : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$ the restriction functor, then the $\mathcal{A}$-modules M of the form $M \cong F_X(N)$, for some (resp. finite-dimensional) $N \in \mathcal{A}^X\text{-Mod}$, are precisely the $\mathcal{A}$-modules M such that its restriction $R^{\mathcal{A}}_{\mathcal{B}}(M)$ is isomorphic in $\mathcal{B}\text{-Mod}$ to a (resp. finite) direct sum of direct summands of X.*
3. *For any $N \in \mathcal{A}^X\text{-Mod}$, we have $\underline{\ell}(F_X(N)) = [X]\underline{\ell}(N)^t$, where $[X]$ denotes the matrix with $[X]_{ij} = \dim_{D_i} e_i X f_j$. Here, $e_1, \ldots, e_n$ and $f_1, \ldots, f_t$ are the primitive central orthogonal idempotents determined by decomposition of units $1 = \sum_i e_i$ in R and $1 = \sum_j f_j$ in S.*
4. *The functor F_X is length controlling, which means that there is a constant $C \in \mathbb{N}$ such that $\ell(N) \leq \ell(F_X(N)) \leq C\ell(N)$, for $N \in \mathcal{A}^X\text{-Mod}$.*

Proof. The finite-dimensional algebra $\Gamma = \mathrm{End}_{\mathcal{B}}(X)^{op}$ admits the splitting $\Gamma = S \oplus P$, where P is the radical of Γ, because k is a perfect field. The semisimple algebra S is basic because the indecomposable direct summands of X are pairwise non-isomorphic. From [8](5.4), we know that $\mathcal{A}$ is a Roiter ditalgebra. Hence, from [8](17.1)–(17-2), the $\mathcal{B}$-module X is admissible. The $\mathcal{B}$-module X is complete by [8](13.3), see [8](12.4), and it is triangular by [8](17.4). Then, $\mathcal{A}^X$ is a triangular ditalgebra, its natural triangular structure is described in [8](14.10).

The proof of the first item can be found in [4], but for completeness, we recall the argument. Make $M := F^X(N)$ and $H^X := \mathrm{Hom}_R(W_0 \otimes_R M, M)$. Then,

$$\begin{aligned} H &:= \mathrm{Hom}_S(X^* \otimes_B BW_0''B \otimes_B X \otimes_S N, N) \\ &\cong \mathrm{Hom}_B(BW_0''B \otimes_B X \otimes_S N, X \otimes_B N) \\ &\cong \mathrm{Hom}_B(BW_0''B \otimes_B M, M) \\ &\cong \mathrm{Hom}_B(B \otimes_R W_0'' \otimes_R B \otimes_B M, M) \\ &\cong \mathrm{Hom}_B(B \otimes_R W_0'' \otimes_R M, M) \\ &\cong \mathrm{Hom}_R(W_0'' \otimes_R M, M), \end{aligned}$$

where the first isomorphism is given by the fact that X is a finitely generated projective right S-module and the third one is due to [8](12.2)(3). Then, $\dim_k H^X = \dim_k \mathrm{Hom}_R(W_0' \otimes_R M, M) + \dim_k H$, and from this we obtain the formula in the statement of the lemma.

Denote by $\{e_i\}_{i=1}^n$, the orthogonal primitive central idempotents given by the unit decomposition of R, assume that M is sincere and make $\underline{\ell} = \underline{\ell}(M)$, then

$$\dim_k \mathrm{Hom}_R(W_0' \otimes_R M, M) = \sum_{i,j} \ell_i \ell_j \dim_k(e_i W_0' e_j),$$

which is clearly not zero if $W_0' \neq 0$.

In order to prove the second item, denote by $\{f_j\}_{j=1}^t$ the orthogonal primitive central idempotents given by the unit decomposition of S. Consider an $\mathcal{A}$-module M. Then, the condition $M \cong X \otimes_S N$ in B-Mod, for some S-module N can be replaced by $M \cong \oplus_{j=1}^t Xf_j \otimes_{Sf_t} f_t N$ in B-Mod, for some S-module N. The last one, since Sf_t is a division algebra and $Xf_j = X_j$, is equivalent to $M \cong \oplus_{j=1}^t X_j^{(I_j)}$ in B-Mod, where each I_j is a basis of the Sf_j-vector space $f_j N$, for some S-module N. But clearly, such an S-module N determines the sets $I_1, \dots, I_t$, and given a family of sets $I_1, \dots, I_t$, we can construct the S-module $N := \oplus_{j=1}^t Sf_j^{(I_j)}$. Then, our statement follows from [8](25.5). Of course, for a finite-dimensional $\mathcal{A}$-module M, all the basis $I_1, \dots, I_t$ are finite.

For $i \in [1, n]$, we have the equalities

$$\ell_{D_i}(e_i F^X(N)) = \ell_{D_i}(e_i X \otimes_S N) = \sum_j \ell_{D_i}(e_i X f_j \otimes_{Sf_j} f_j N) = \sum_j [X]_{ij} \ell_{Sf_j}(f_j N),$$

and item 3 follows. Adding over $i \in [1, n]$, we obtain $\ell(F^X(N)) = \sum_{ij}[X]_{ij}\ell_{Sf_j}(f_j N) \geq \sum_j \ell_{Sf_j}(f_j N) = \ell(N)$. Item 4 follows from this. $\square$

Lemma 7.4. *Given an admissible ditalgebra $\mathcal{A}$, assume that the admissible ditalgebra $\mathcal{A}'$ is obtained from $\mathcal{A}$ by a finite sequence of reductions of either of the types described in (2.5), (2.6) or (2.7) and that B' is a generically tame minimal algebra of infinite representation type, which is an initial subalgebra of $\mathcal{A}'$. Consider the composite functor*

$$B'\text{-Mod} \xrightarrow{E} \mathcal{A}'\text{-Mod} \xrightarrow{G} \mathcal{A}\text{-Mod},$$

where E is the associated extension functor and G is the composition of the reduction functors associated to the finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}'$. Then, we have the following.

1. *If the functor GE maps one indecomposable regular B'-module onto a sincere $\mathcal{A}$-module, it maps each indecomposable regular B'-module onto a sincere $\mathcal{A}$-module.*
2. *For any $C \in \mathbb{N}$, almost every non-regular finite-dimensional indecomposable B'-module N with $GE(N)$ sincere satisfies that $\|GE(N)\| > C$.*

Proof. (1) Notice that, since B' is a proper subalgebra of $\mathcal{A}'$, the functor E preserves length vectors, that is $\underline{\ell}(H) = \underline{\ell}(E(H))$, for each $H \in B'$-mod.

We consider first the case where $\mathcal{A}' = \mathcal{A}$, thus G does not appear. Assume that $H \in B'$-Mod is such that $E(H)$ is a sincere $\mathcal{A}'$-module. If B' is infinite-dimensional, then the length vector of any non-trivial B'-module H has only one component, which is not zero, and the same holds for $E(H)$. Thus, the first item is trivial in this case (where we agreed to call regular indecomposable any finite-dimensional indecomposable B'-module). If B' is finite-dimensional, since it has infinite representation type, it makes sense to consider the regular B'-modules. As remarked in (6.6), there is a vector $\underline{\lambda}$, such that for any indecomposable regular B-module H, we have $\underline{\ell}(H) = c_H\underline{\lambda}$, for some $c_H \in \mathbb{N}$. Then, the fact that $\underline{\ell}(E(H)) = c_H\underline{\lambda}$ has no zero components, that is $E(H)$ is sincere, occurs simultaneously for all indecomposable regular B'-modules H or for none of them.

Now, we look at the case where there is a functor G. Again, we use the existence of a vector $\underline{\lambda}$ such that, for any indecomposable regular B'-module H, we have $\underline{\ell}(H) = c_H\underline{\lambda}$, for some $c_H \in \mathbb{N}$. Notice that this fact also holds for the case of an infinite-dimensional B'. Having in mind (7.3)(3), we have a matrix Q such that

$$\underline{\ell}(GE(H)) = Q\underline{\ell}(E(H))^t = Q\underline{\ell}(H)^t = c_H Q(\underline{\lambda})^t.$$

Assume that $GE(H_0)$ is sincere, for some indecomposable regular B'-module H_0. Applying the last equality to H_0, we get that $Q(\underline{\lambda})^t$ has no zero component. Then, the same equality shows that $\underline{\ell}(GE(H))$ has no zero component for any indecomposable regular B'-module H. Thus, any such $GE(H)$ is sincere.

(2) If B' is infinite-dimensional, there is nothing to show: every finite-dimensional indecomposable B'-module is regular. Thus assume that B' is finite-dimensional. Then, as we already mentioned in (6.6), for any $C \in \mathbb{N}$, there are only finitely many pairwise non-isomorphic preprojective or preinjective indecomposable B'-modules with length bounded by C. Then, in case $\mathcal{A} = \mathcal{A}'$, item 2 follows from (7.2).

In the general case, assume that there are infinitely many non-isomorphic non-regular indecomposable B'-modules N such that their image $GE(N)$ is sincere indecomposable with $\|GE(N)\| \leq C$. Then, from (7.2), there are infinitely many non-isomorphic non-regular indecomposable B'-modules N such that their image $GE(N)$ is sincere indecomposable with $\ell(GE(N)) \leq nC$. Since reduction functors are length controlling, see (7.2)(3), these modules N satisfy $\ell(N) \leq \ell(GE(N)) \leq nC$, contradicting (6.6). $\square$

Theorem 7.5. *Assume that the admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over the infinite perfect field k. Then, for any integer $d \geq 0$, there are constructible ditalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m$, and pregenerically tame minimal algebras of infinite representation type $B_1, \dots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and a family of functors $F_1, \dots, F_m$ such that*

1. *the functor $F_i : B_i$-Mod $\longrightarrow \mathcal{A}$-Mod preserves indecomposability and isomorphism classes, for any $i \in [1, m]$;*
2. *for almost every sincere indecomposable $M \in \mathcal{A}$-Mod with $\|M\| \leq d$ there exist $i \in [1, m]$ and $N \in B_i$-mod such that $F_i(N) \cong M$ in $\mathcal{A}$-Mod;*
3. *the functor $F_i : B_i$-mod $\longrightarrow \mathcal{A}$-mod maps regular B_i-modules onto sincere $\mathcal{A}$-modules, for $i \in [1, m]$;*
4. *if $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $F_i(N_u) \cong F_j(M_u)$ for all $u \in U$, then $i = j$;*
5. *each functor F_i is the composition $B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{G_i} \mathcal{A}\text{-Mod}$, where E_i is the associated extension functor and G_i is the composition of the reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ to $\mathcal{A}_i$.*

Proof. Since $\mathcal{A}$ is constructible, from (4.6), we know that $\mathcal{A}$ is pregenerically tame. The same will remain true for any ditalgebra obtained from $\mathcal{A}$ by a finite number of the permitted reductions.

Given $d \geq 0$, we say that an admissible ditalgebra $\mathcal{A}'$ has *finite d-representation type* iff there is only a finite number of isoclasses of indecomposable $\mathcal{A}'$-modules M with $\|M\| \leq d$. We say that $\mathcal{A}'$ has *finite sincere d-representation type* iff there is only a finite number of isoclasses of sincere indecomposable $\mathcal{A}'$-modules M with $\|M\| \leq d$.

We shall prove the theorem by induction on d. If $d = 0$ and $M \in \mathcal{A}$-Mod is sincere with $\|M\| = 0$ then, the layer of $\mathcal{A}$ is of the form (R, W), with $W_0 = 0$. Hence, $\mathcal{A}$ has finite representation type and there is nothing to show. So assume that $d > 0$ and that the theorem holds for any admissible ditalgebra $\mathcal{A}'$ constructible from a generically tame finite-dimensional basic algebra and any $d' < d$.

Now, we have to consider the sincere indecomposable modules $M \in \mathcal{A}$-Mod with $\|M\| \leq d$. If $\mathcal{A}$ admits only finitely many isoclasses of such modules, we have nothing to show (there are no such families of functors). So we assume that $\mathcal{A}$ is of infinite sincere d-representation type.

Since $\mathcal{A}$ is an admissible ditalgebra, we can look at the triangular filtration $0 = W_0^0 \subseteq W_0^1 \subseteq \cdots \subseteq W_0^s = W_0$, which is additive, as in [8](5.1), because the field k is perfect and hence $R \otimes_k R$ is semisimple. Then, after performing a refinement, if necessary, we can assume that W_0^1 is a simple direct summand of the R–R-bimodule W_0. Then, by triangularity, we have that $\delta(W_0^1) \subseteq W_1$. Moreover, we have R–R-bimodule decompositions $W_0 = W_0^1 \oplus W_0''$ and $W_1 = \delta(W_0^1) \oplus W_1''$. We consider two cases.

Case 1: $\delta(W_0^1) \neq 0$.

Since W_0 is a simple R–R-bimodule, $W_0^1 \cap \operatorname{Ker} \delta = 0$ and we can apply the regularization described in (2.6), to obtain a length preserving equivalence $F_r : \mathcal{A}^r\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, as in [8](8.19). If $\mathcal{A}^r$ is of finite sincere $(d-1)$-representation type, then $\mathcal{A}$ is of finite sincere d-representation type. Indeed, given any sincere indecomposable $\mathcal{A}$-module with $\|M\| = d$, there is $N \in \mathcal{A}^r$-Mod with $F^r(N) \cong M$ and $\|N\| < \|M\|$, and so N is a sincere $\mathcal{A}^r$-module with $\|N\| \leq d-1$. Thus, if there are only finitely many possible isoclasses of such $\mathcal{A}^r$-modules N, there will be only finitely many possible isoclasses of such $\mathcal{A}$-modules M.

By assumption, $\mathcal{A}$ is of infinite sincere d-representation type, thus $\mathcal{A}^r$ is of infinite sincere $(d-1)$-representation type and we can apply the induction hypothesis to the constructible pregenerically tame ditalgebra $\mathcal{A}^r$ and $d-1$, to obtain a family of functors $F_i : B_i\text{-Mod} \longrightarrow \mathcal{A}^r\text{-Mod}$, $i \in [1, m]$, satisfying the corresponding conditions 1–5. Let us show that the family $\mathcal{F} := \{F^r F_i \mid i \in [1, m]\}$ is the required family of functors for $\mathcal{A}$ and d.

Item 1 is clear because F^r preserves indecomposables and isomorphism classes. Hence, so does every functor in $\mathcal{F}$. Item 2, is also clear, since we realize every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ as $F^r(N) \cong M$, for some $\mathcal{A}^r$-module N with $\|N\| < d$. Then, we can apply our induction hypothesis to almost every such indecomposable sincere $\mathcal{A}^r$-module N to obtain $F_i(H) \cong N$, for some $H \in B_i$-mod, thus $M \cong F^r F_i(N)$ and we are done. Item 3, follows from the induction hypothesis and the fact that the functor F^r maps sincere modules onto sincere modules. Finally, item 4 follows from the induction hypothesis and the fact that F^r reflects isomorphisms.

Case 2: $\delta(W_0^1) = 0$.

Consider the initial subalgebra B of $\mathcal{A}$ determined by the R–R-bimodule W_0^1 given above. Consider also the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ and the restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$. Let us first examine this algebra B.

Since W_0^1 is a simple R–R-bimodule, there exist $i, j \in [1, n]$ with $e_j W_0^1 e_i = W_0^1$. If $i = j$, for notational simplicity, we assume that $i = 1$ and write $e := e_1$. Then, $B = T_R(W_0^1) \cong T_{D_1}(W_0^1) \times D_2 \times \cdots \times D_n$. Thus $eBe = T_{D_1}(W_0^1)$ is a minimal algebra. If $i \neq j$, for notational simplicity, we assume that $i = 1$, $j = 2$ and write $e = e_1 + e_2$. Then, $B = T_R(W_0^1) \cong T_{D_1 \times D_2}(W_0^1) \times D_3 \times \cdots \times D_n$. Then, $eBe = T_{D_1 \times D_2}(W_0^1)$ is a minimal algebra.

In both cases, we have that the full and faithful extension functor $Be \otimes_{eBe} - : eBe\text{-Mod} \longrightarrow B\text{-Mod}$ is such that with only finitely many possible exceptions, the isoclasses of the indecomposable B-modules are represented by modules of the form $Be \otimes_{eBe} H$, for some indecomposable $H \in eBe$-Mod.

Since $\mathcal{A}$ is pregenerically tame, from (2.4), we know that B is also pregenerically tame. It follows that eBe is pregenerically tame too. We can consider the composition functor F

$$eBe\text{-Mod} \xrightarrow{Be\otimes_{eBe}-} B\text{-Mod} \xrightarrow{E} \mathcal{A}\text{-Mod},$$

which, from (2.4), preserves indecomposability and isomorphism classes. Notice that if F maps an indecomposable eBe-module onto a sincere $\mathcal{A}$-module, then $B = eBe$.

In case the algebra B is not of finite representation type, apply (5.2) and (5.3) to the number nd to obtain a finite family $\mathcal{I}(nd)$ of finite-dimensional indecomposable B-modules such that, for any indecomposable $\mathcal{A}$-module M with $\ell(M) \leq nd$ and $M \ncong E(N)$ in $\mathcal{A}$-Mod, for any $N \in B$-Mod, the module $R(M)$ is isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(nd)$.

If we are in the case where the algebra B is of finite representation type, we denote by $\mathcal{I}(nd)$ a complete set of non-isomorphic finite-dimensional indecomposable B-modules, to obtain trivially that, for any indecomposable $\mathcal{A}$-module M with $\ell(M) \leq nd$ in $\mathcal{A}$-Mod, we have that $R(M)$ is isomorphic in B-Mod to a direct sum of modules of $\mathcal{I}(nd)$.

In any case, B of finite representation type or not, let $X_1, \ldots, X_t$ be a complete set of pairwise non-isomorphic representatives of the B-modules in $\mathcal{I}(nd)$ and make $X := X_1 \oplus \cdots \oplus X_t$. Consider the reduction $\mathcal{A} \mapsto \mathcal{A}^X$ described in (7.3) and its associated functor $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. From (2.7), the ditalgebra $\mathcal{A}^X$ is pregenerically tame and constructible. The class of indecomposable $\mathcal{A}$-modules of length $\leq nd$ is contained in the union of the classes $\operatorname{Im} E$ and $\operatorname{Im} F_X$, where $\operatorname{Im} E$ denotes the class of $\mathcal{A}$-modules of the form $M \cong E(N)$, for some $N \in B$-Mod, and $\operatorname{Im} F_X$ denotes the indecomposable $\mathcal{A}$-modules M, which satisfy that $R(M) \cong m_1X_1 \oplus \cdots \oplus m_tX_t$ in B-Mod, for some $m_1, \ldots, m_t \geq 0$. It follows, from (7.3)(2), that a finite-dimensional $\mathcal{A}$-module M with the last property has the form $F_X(N) \cong M$, for some $N \in \mathcal{A}^X$-mod. From the previous discussion and (7.2), any sincere indecomposable $\mathcal{A}$-module with $\|M\| \leq d$ lies in $\operatorname{Im} E \bigcup \operatorname{Im} F_X$.

Assume that $\mathcal{A}^X$ is of finite $(d-1)$-representation type. Hence, there can only be finitely many isoclasses of sincere indecomposable $\mathcal{A}$-modules $M \in \operatorname{Im} F_X$ with $\|M\| \leq d$. Then, almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ lies in $\operatorname{Im} E$, and there is an infinite number of pairwise non-isomorphic such modules M, because we are assuming that $\mathcal{A}$ has infinite sincere d-representation type. Let us see that in this case, the family $\mathcal{F} := \{F\}$ satisfies properties 1–4 for $\mathcal{A}$ and d.

We already know that F satisfies item 1. Recall that we are assuming that almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ lies in $\operatorname{Im} E$ and that they determine infinitely many isoclasses. Then, having in mind that almost every indecomposable B-module is an eBe-module, we get that for almost every such M there is an indecomposable eBe-module H with $F(H) \cong M$. Thus, $B = eBe$ and $F \cong E$. Moreover, from (7.4)(2), almost every such H is a regular B-module. Thus, item 2 holds. From (7.4)(1), the functor F maps indecomposable regular modules onto sincere $\mathcal{A}$-modules (item 3). Finally, item 4 holds trivially because there is only one functor in $\mathcal{F}$.

From now on, we assume that $\mathcal{A}^X$ has infinite $(d-1)$-representation type.

Consider the constructible ditalgebras $\mathcal{A}^{Xd_1}, \ldots, \mathcal{A}^{Xd_t}$ obtained from $\mathcal{A}^X$ by deletion of a finite number of idempotents of S, and the corresponding reduction functors $F^{d_i} : \mathcal{A}^{Xd_i}\text{-Mod} \longrightarrow \mathcal{A}^X\text{-Mod}$, for $i \in [1, t]$.

Since $\mathcal{A}^X$ has infinite $(d-1)$-representation type, it admits an infinite family of pairwise non-isomorphic indecomposable modules N with $\|N\| \leq d-1$. This infinite family determines either an infinite family of sincere $\mathcal{A}^X$-modules with norm $\leq d-1$, thus $\mathcal{A}^X$ has infinite sincere $(d-1)$-representation type, or there is a finite subset of idempotents of S such that the ditalgebra

$\mathcal{A}^{Xd_i}$ obtained from $\mathcal{A}^X$ by eliminating these idempotents admits an infinite family of sincere indecomposable $\mathcal{A}^{Xd_i}$-modules with norm $\leq d-1$.

Define $\mathcal{A}^{Xd_0} := \mathcal{A}^X$ and denote by $F^{d_0} : \mathcal{A}^{Xd_0}\text{-Mod} \longrightarrow \mathcal{A}^X\text{-Mod}$ the identity functor. Every sincere $\mathcal{A}^X$-module M lies in $\mathcal{A}^{Xd_0}$-Mod and $F^{d_0}(M) = M$.

Now, we consider the subset I of $[0,t]$ defined by $i \in I$ iff the ditalgebra $\mathcal{A}^{Xd_i}$ is of infinite sincere $(d-1)$-representation type, and discard the remaining ones. The above discussion shows that $I \neq \emptyset$.

If M is a sincere indecomposable $\mathcal{A}$-module with $\|M\| \leq d$, such that for some $N \in \mathcal{A}^X$-Mod, we have $F^X(N) \cong M$, then there is a sincere indecomposable $\mathcal{A}^{Xd_j}$-module L with $F^{d_j}(L) \cong N$, hence $\|L\| = \|N\| < \|M\| \leq d$. Thus, there are only finitely many possible choices for such L, when $\mathcal{A}^{Xd_j}$ is of finite sincere $(d-1)$-representation type. Thus, eliminating all the $\mathcal{A}^{Xd_j}$ of sincere finite $(d-1)$-representation type, we will only loose a finite number of isoclasses of such modules M.

Then, apply the induction hypothesis to each $\mathcal{A}^{Xd_i}$ and $d-1$, for $i \in I$, to obtain minimal algebras $\{B_{ij}\}_{j=1}^{n_i}$ and functors $\{F_{ij} : B_{ij}\text{-Mod} \longrightarrow \mathcal{A}^{Xd_i}\text{-Mod}\}_{j=1}^{n_i}$ satisfying the corresponding requirements. Then, for any i and j, we can consider the compositions

$$B_{ij}\text{-Mod} \xrightarrow{F_{ij}} \mathcal{A}^{Xd_i}\text{-Mod} \xrightarrow{F_{d_i}} \mathcal{A}^X\text{-Mod} \xrightarrow{F_X} \mathcal{A}\text{-Mod}.$$

We will extract the family of functors we need for $\mathcal{A}$ and d from the family

$$\mathcal{F} := \{F\} \cup \{F^X F^{d_i} F_{ij} \mid i \in I \text{ and } j \in [1, n_i]\}.$$

First, we show that the functors in the family $\mathcal{F}$ cover almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$. That is item 2 is satisfied by this family. Indeed, given such a module M, by the discussion above, we have that $M \in \operatorname{Im} F_X \bigcup \operatorname{Im} E$.

For almost all $M \in \operatorname{Im} F_X$, we have that $M \cong F_X(N)$ for some $N \in \mathcal{A}^X$-mod and $N \cong F^{d_i}(L)$, for some $i \in I$ and some sincere indecomposable $L \in \mathcal{A}^{Xd_i}$-mod. We know that $\|L\| = \|N\| < \|M\| \leq d$, because M is sincere. Therefore, by induction hypothesis, for almost every such module L, we get $L \cong F_{ij}(H)$ for some $H \in B_{ij}$-mod. Hence, $F^X F^{d_i} F_{ij}(H) \cong M$, as claimed.

If $M \in \operatorname{Im} E$, thus $M \cong E(N)$, for some indecomposable $N \in B$-Mod. Since N is indecomposable and M is sincere, we have $B = eBe$, and $F(N) \cong M$.

In the following discussion we will discard some functors of the family $\mathcal{F}$, without spoiling the covering condition we have just proved for $\mathcal{F}$.

First, if B is of finite representation type then any finite-dimensional indecomposable $M \in \mathcal{A}$-Mod has $R(M)$ isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(nd)$, therefore $M \in \operatorname{Im} F^X$. Then, as we have just seen, almost every sincere indecomposable $\mathcal{A}$-module M in $\operatorname{Im} F_X$ with $\|M\| \leq d$ has the form $F^X F^{d_i} F_{ij}(H) \cong M$, for some $H \in B_{ij}$-mod. Thus we can discard the functor F from the family $\mathcal{F}$.

So, we will assume that the functor F is left in $\mathcal{F}$ only if B has infinite representation type. Then, the algebra eBe, as well as any of the minimal algebras B_{ij} are pregenerically tame of infinite representation type.

If the functor F maps one regular eBe-module H onto a non-sincere $\mathcal{A}$-module, from our previous Lemma 7.4, we know that it maps any regular module onto a non-sincere $\mathcal{A}$-module, and in this case again we can discard the functor F from the family $\mathcal{F}$. Doing this, we only omit to cover the sincere indecomposable $\mathcal{A}$-modules M with norm $\leq d$ which were of the form $F(H) \cong M$, for some non-regular indecomposable eBe-module H (a finite number of isoclasses, according to the last statement of (7.4)).

So, we will assume that F is left in the family $\mathcal{F}$ only if eBe is such that F maps regular eBe-modules onto sincere $\mathcal{A}$-modules.

Now, assume that the functor $F^X F^{d_i} F_{ij}$ is such that $F^X F^{d_i} F_{ij}(H)$ is not sincere, for some indecomposable regular B_{ij}-module H. From (7.4), the functor $F^X F^{d_i} F_{ij}$ covers only finitely many isoclasses of indecomposable sincere $\mathcal{A}$-modules with $\|M\| \leq d$. Hence, we can discard the functor $F^X F^{d_i} F_{ij}$ from our family $\mathcal{F}$, since we only leave aside finitely many possible isoclasses of sincere indecomposable $\mathcal{A}$-modules M with $\|M\| \leq d$ which are may be not covered by the other functors of the family $\mathcal{F}$.

So, we assume that the functor $F^X F^{d_i} F_{ij}$ appears in the family $\mathcal{F}$ only if it maps indecomposable regular modules onto sincere ones.

Now, we have to show that the family $\mathcal{F}$, after discarding the functors pointed out above, satisfies items 1–4. We already know that item 1 holds, because each functor in $\mathcal{F}$ is either a composition of reduction functors or it is F. Item 2 holds, because we only discarded functors when we could cover almost every sincere indecomposable $\mathcal{A}$-module M such that $\|M\| \leq d$ with the remaining functors in $\mathcal{F}$. Item 3 holds, because we discarded every functor in $\mathcal{F}$ without this property. In the following, we proceed to the proof of item 4.

Notice first that, as a consequence of (7.3)(2), if $L \in B$-Mod is indecomposable such that $E(L) \cong F^X(L')$, for some $L' \in \mathcal{A}^X$-Mod, then $L \cong RE(L)$ has to be isomorphic to one of the indecomposable B-modules $X_1, \ldots, X_n$. Thus, for almost every indecomposable $N \in B$-Mod, there is no $L' \in \mathcal{A}^X$-Mod with $F^X(L') \cong E(N)$. This implies that there is no pair of infinite families of pairwise non-isomorphic indecomposables $\{N_u\}_{u\in U}$ in B-Mod and $\{M_u\}_{u\in U}$ in B_{ij}-Mod such that $F^X F^{d_i} F_{ij}(M_u) \cong F(N_u)$, for all $u \in U$.

Assume then that there is a pair of infinite families of pairwise non-isomorphic indecomposable regular modules $\{M_u\}_{u\in U}$ in B_{ij}-mod and $\{N_u\}_{u\in U}$ in $B_{i'j'}$-mod such that $F^X F^{d_{i'}} F_{i'j'}(N_u) \cong F^X F^{d_i} F_{ij}(M_u)$, for all $u \in U$. Then, since F^X reflects isomorphisms, we get $F^{d_{i'}} F_{i'j'}(N_u) \cong F^{d_i} F_{ij}(M_u)$, for all $u \in U$. In particular, they have the same support. But item 3 holds for the families $\{F_{ij}\}_j$ and $\{F_{i'j'}\}_{j'}$ and so $F_{i'j'}(N_u)$ and $F_{ij}(M_u)$ are sincere modules over $\mathcal{A}^{Xd_{i'}}$ and $\mathcal{A}^{Xd_i}$, respectively, hence $i = i'$. Then, $F_{ij'}(N_u) \cong F_{ij}(M_u)$, for all $u \in U$. From the induction hypothesis, $j = j'$.

Then, the family of functors $\mathcal{F}$ is what we wanted to construct, the last item follows from the given construction. □

Theorem 7.6. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over the infinite perfect field k. Then, for any integer $d \geq 0$, there are constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$, and pregenerically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and a family of functors $F_1, \ldots, F_m$ such that*

1. *the functor $F_i : B_i$-Mod $\longrightarrow \mathcal{A}$-Mod preserves indecomposability and isomorphism classes, for any $i \in [1, m]$;*
2. *for almost every indecomposable $M \in \mathcal{A}$-Mod with length $\leq d$ there exist $i \in [1, m]$ and $N \in B_i$-mod such that $F_i(N) \cong M$ in $\mathcal{A}$-Mod;*
3. *if $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $F_i(N_u) \cong F_j(M_u)$ for all $u \in U$, then $i = j$;*
4. *each functor F_i is the composition B_i-Mod $\xrightarrow{E_i} \mathcal{A}_i$-Mod $\xrightarrow{G_i} \mathcal{A}$-Mod, where E_i is the associated extension functor and G_i is the composition of the reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}_i$.*

Proof. From (4.6), we know that $\mathcal{A}$ is pregenerically tame. Notice that, in order to prove our Theorem 7.6, it will be enough to prove items 1, 3, 4 and the following item $2'$.

$2'$. *For almost every indecomposable* $M \in \mathcal{A}$-Mod *with norm* $\leq d$ *there are* $i \in [1, m]$ *and* $N \in \mathcal{B}_i$-mod *with* $F_i(N) \cong M$ *in* $\mathcal{A}$-Mod.

Indeed, given $d \geq 0$, there are only finitely many length vectors $\underline{\ell}$ such that $\sum_{i=1}^{n} \ell_i \leq d$, consider their maximal norm $\hat{d} := \max_{\underline{\ell}}\{\|\underline{\ell}\|\}$. Then, if $2'$ holds for $\hat{d}$, any indecomposable $\mathcal{A}$-module M with length $\leq d$ has length vector $\underline{\ell} := \underline{\ell}(M)$ satisfying $\sum_{i=1}^{n} \ell_i \leq d$ and, therefore, $\|M\| \leq \hat{d}$ and we can apply $2'$ to obtain 2.

We assume that $\mathcal{A}$ has infinite d-representation type, otherwise, there is nothing to prove.

Consider the admissible constructible ditalgebras $\mathcal{A}^{d_1}, \ldots, \mathcal{A}^{d_t}$ obtained from $\mathcal{A}$ by deletion of a finite number of idempotents of R. We consider also $\mathcal{A}^{d_0} := \mathcal{A}$ and the identity functor $F^{d_0} : \mathcal{A}^{d_0}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, thus every sincere indecomposable $\mathcal{A}$-module M lies in $\mathcal{A}^{d_0}$-Mod and $F^{d_0}(M) = M$. Consider the subset I of $[0, t]$ defined by $i \in I$ iff $\mathcal{A}^{d_i}$ is of infinite d-representation type. Then, apply (7.5) to each $\mathcal{A}^{d_i}$ and d, for $i \in I$, to obtain minimal algebras $\{B_{ij}\}_{j=1}^{n_i}$ and functors $F_{ij} : B_{ij}\text{-Mod} \longrightarrow \mathcal{A}^{d_i}\text{-Mod}$ satisfying the corresponding statements 1–5 of (7.5) for each $\mathcal{A}^{d_i}$ and d. Then, we can consider the family of compositions

$$\mathcal{F} := \{B_{ij}\text{-Mod} \xrightarrow{F_{ij}} \mathcal{A}^{d_i}\text{-Mod} \xrightarrow{F_{d_i}} \mathcal{A}\text{-Mod} \mid i \in I \text{ and } j \in [1, n_i]\}.$$

It is clear that the family $\mathcal{F}$ satisfies item 1, because the families $\{F_{ij}\}_j$ do so and F^{d_i} preserves indecomposables and isomorphism classes. The family $\mathcal{F}$ also satisfies $2'$ because, given any indecomposable $M \in \mathcal{A}$-Mod with $\|M\| \leq d$, we have $M \cong F_{d_i}(N)$, for some sincere indecomposable $N \in \mathcal{A}^{d_i}$-Mod with $\|N\| = \|M\| \leq d$. For almost all of these indecomposables N, we have $F_{ij}(H) \cong N$, for some $H \in B_{ij}$-mod.

Assume then that there is a pair of infinite families of pairwise non-isomorphic indecomposable regular modules $\{N_u\}_{u \in U}$ in B_{ij}-mod and $\{M_u\}_{u \in U}$ in $B_{i'j'}$-mod such that $F^{d_{i'}} F_{i'j'}(M_u) \cong F^{d_i} F_{ij}(N_u)$, for all $u \in U$. Then, proceeding as in the proof of last theorem, we obtain $i = i'$ and $j = j'$.

Finally, item 4 is also satisfied because each F_{ij} satisfies (7.5)(5). □

Remark 7.7. Given an admissible ditalgebra $\mathcal{A}$ with layer (R, W), there is a constant $c \in \mathbb{N}$ such that, $\ell(M) \leq \dim_k M \leq c\ell(M)$, for any $M \in \mathcal{A}$-Mod. Therefore, in the statement of the last theorem, we can replace the phrase *with length* $\leq d$ by the phrase *with k-dimension* $\leq d$, and obtain a valid result.

8. Transition to finite-dimensional algebras

Lemma 8.1. *Assume that* B *is a proper subalgebra of the admissible ditalgebra* $\mathcal{A}$ *and denote by* $E : B\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, *the extension functor. Consider the natural exact structure on* $\mathcal{A}$-Mod *(see* [8](6.8)*). Then, the functor* E *is exact and induces an injective morphism* $\mathrm{Ext}_B(-, ?) \longrightarrow \mathrm{Ext}_{\mathcal{A}}(E(-), E(?))$.

Proof. Clearly, whenever we have an exact sequence $0 \longrightarrow N \xrightarrow{f^0} L \xrightarrow{g^0} M \longrightarrow 0$ in B-Mod, we have in A-Mod the exact sequence

$$0 \longrightarrow E(N) \xrightarrow{f^0} E(L) \xrightarrow{g^0} E(M) \longrightarrow 0$$

and, hence, the exact pair $E(N) \xrightarrow{(f^0,0)} E(L) \xrightarrow{(g^0,0)} E(M)$ in the exact category $\mathcal{A}$-Mod; see [8](6.2). Thus, the functor E is exact. If the previous conflation splits, there is $(h^0, h^1) \in \mathrm{Hom}_{\mathcal{A}}(E(L), E(N))$ with $(h^0, h^1)(f^0, 0) = I_{E(N)}$. In particular, $h^0 f^0 = Id$. Recall that, given the layer (R, W) of $\mathcal{A}$, the algebra B has the form $B = T_R(W_0')$, for some direct summand W_0' of W_0 with $\delta(W_0') = 0$. It follows that h^0 is a morphism of B-modules and hence the original sequence splits. Then, the functor E induces an injection $\mathrm{Ext}_B(-, ?) \longrightarrow \mathrm{Ext}_{\mathcal{A}}(E(-), E(?))$. □

Theorem 8.2. *Let Λ be a generically tame finite-dimensional algebra over an infinite perfect field k and let d be a non-negative integer. Then, there is a finite sequence of pregenerically tame minimal algebras of infinite representation type $B_1, \dots, B_m$, and Λ–B_i-bimodules $Z_1, \dots, Z_m$, which are finitely generated as right B_i-modules, satisfying the following.*

1. *The functor $Z_i \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes of modules without injective direct summands, if B_i is a minimal algebra of the first type in (6.1).*
2. *The functor $Z_i \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes, if B_i is a minimal algebra of the second type in (6.1).*
3. *Almost every indecomposable Λ-module M with $\dim_k M \leq d$ is isomorphic to $Z_i \otimes_{B_i} N$, for some $i \in [1, m]$ and some $N \in B_i\text{-mod}$.*
4. *If $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $Z_i \otimes_{B_i} N_u \cong Z_j \otimes_{B_j} M_u$ for all $u \in U$, then $i = j$.*

Proof. We first show that we can assume that Λ is a basic algebra. Indeed, assume that the theorem holds for basic algebras, and assume that Λ is not basic and take $d \geq 0$. Consider a basic finite-dimensional k-algebra Λ' which is Morita equivalent to Λ. From [8](29.8)(1), we know that Λ' is generically tame. Consider a Λ–Λ'-bimodule P, finitely generated projective by the right which realizes the equivalence $P \otimes_{\Lambda'} - : \Lambda'\text{-Mod} \longrightarrow \Lambda\text{-Mod}$. Thus, for instance from [8](27.11), there is some $s \in \mathbb{N}$ such that for any $M' \in \Lambda'$-mod, we have that $\dim_k M' \leq s \times \dim_k(P \otimes_{\Lambda'} M')$. Define $d' := s \times d$. Then, by assumption, there are minimal algebras $B_1, \dots, B_m$ and Λ'–B_i-bimodules $Z_1', \dots, Z_m'$, which are finitely generated as right B_i-modules, satisfying the following.

1′. *The functor $Z_i' \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda'\text{-Mod}$ preserves indecomposability and isomorphism classes of modules without injective direct summands, if B_i is a minimal algebra of the first type.*

2′. *The functor $Z_i' \otimes_{B_i} - : B_i\text{-Mod} \longrightarrow \Lambda'\text{-Mod}$ preserves indecomposability and isomorphism classes, if B_i is a minimal algebra of the second type.*

3′. *Almost every indecomposable Λ'-module M' with $\dim_k M' \leq d'$ is isomorphic to $Z_i' \otimes_{B_i} N$, for some $i \in [1, m]$ and some $N \in B_i\text{-mod}$.*

4′. *If $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $Z_i' \otimes_{B_i} N_u \cong Z_j' \otimes_{B_j} M_u$ for all $u \in U$, then $i = j$.*

Consider, for each i, the Λ–B_i-bimodule $Z_i := P \otimes_{\Lambda'} Z_i'$. Since P is a finitely generated projective right Λ'-module, each bimodule Z_i is finitely generated as a right B_i-module. Let us show that items 1– 4 hold for Λ.

Since $P \otimes_{\Lambda'} Z_i' \otimes_{B_i} -$ is the composition of $Z_i' \otimes_{B_i} -$ and the equivalence $P \otimes_{\Lambda'} -$, the first and second items are clear. For the third item, take an indecomposable $M \in \Lambda$-Mod with

$\dim_k M \leq d$ and take $M' \in \Lambda'$-Mod with $P \otimes_{\Lambda'} M' \cong M$. Then, as we mentioned above, $\dim_k M' \leq s \times d = d'$ and, for almost all such M', we have $Z'_i \otimes_{B_i} N \cong M'$, for some i and some B_i-module N. Therefore, $Z \otimes_{B_i} N \cong M$, and we are done.

From now on, we assume that our given finite-dimensional algebra Λ is basic. Since k is a perfect field, the algebra Λ splits over its radical J and, therefore, its Drozd's ditalgebra $\mathcal{D} := \mathcal{D}^{\Lambda}$ is admissible. By definition, $\mathcal{D}$ is constructible from the generically tame algebra Λ. Apply (7.6) to the ditalgebra $\mathcal{D}$ and the integer $d' := (1 + \dim_k \Lambda) \dim_k \Lambda \times d$ to obtain the corresponding constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and pregenerically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and the corresponding family of functors $F_i : B_i\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$ such that (7.6)(1)–(4) hold for $\mathcal{D}$ and d'.

For a fixed $i \in [1, m]$, adopt the notation $\mathcal{A} = \mathcal{A}_i = \mathcal{D}^{z_1 \cdots z_n}$ and $B = B_i$. Then, we have that the functor F_i is isomorphic to the composition:

$$B\text{-Mod} \xrightarrow{B \otimes_B -} B\text{-Mod} \xrightarrow{E} \mathcal{D}^{z_1 \cdots z_n}\text{-Mod} \xrightarrow{G = F^{z_1} \cdots F^{z_n}} \mathcal{D}\text{-Mod}.$$

The B–B-bimodule B is free finitely generated by the right; hence the A–B-bimodule $E(B)$ is free finitely generated by the right. We have the equality of functors $L_{\mathcal{A}}(E(B) \otimes_B -) = E(B \otimes_B -)$ and we can apply [8](22.7), to obtain that $GE(B)$ is a D–B-bimodule projective by the right and the composition of the functor $E(B) \otimes_B -$ with the restriction $A\text{-Mod} \longrightarrow D\text{-Mod}$ of G is given by the tensor $G(E(B)) \otimes_B -$. Here, $D = A_{\mathcal{D}}$ and $A = A_{\mathcal{A}}$. Notice that $F_i = L_{\mathcal{D}}(G(E(B)) \otimes_B -)$ and recall that it preserves isomorphism classes and indecomposables. Consider the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$.

Claim 1. *If B_i is a minimal algebra of the second type, then for any $M \in B_i$-Mod, we have $\Xi_\Lambda F_i(M) \in \mathcal{P}^2(\Lambda)$.*

Proof of Claim 1. We use the fact that projective finitely generated B_i-modules are free of finite rank. Indeed, from Cohn's theorem (see [9,10]), B_i is a free ideal ring; from Kaplanski's theorem (see [29]), every projective B_i-module is free; finally, B_i has the IBN property, because it admits a finite-dimensional algebra as a quotient.

We have in mind also that there is an infinite family of non-isomorphic finite-dimensional indecomposable B_i-modules.

Then, we can apply [8](22.20), to derive that no non-zero object of the form $(P, 0, 0)$ is isomorphic in $\mathcal{P}^1(\Lambda)$ to one of the form $F_i(M)$, with $M \in B_i$-Mod. This ends the proof of Claim 1. □

Claim 2. *If B_i is a minimal algebra of the first type then, for any non-injective indecomposable $M \in B_i$-Mod, we have $\Xi_\Lambda F_i(M) \in \mathcal{P}^2(\Lambda)$.*

Proof of Claim 2. We use the natural exact structures on the module category of a Roiter ditalgebra and on $\mathcal{P}^1(\Lambda)$ defined in [8]Section 6(6.8) and [8]Section 18. Relative to the exact structure in $\mathcal{P}^1(\Lambda)$, from [8](18.3), the objects $(P, 0, 0)$ are injective. Since the indecomposable objects in $\mathcal{P}^1(\Lambda) \setminus \mathcal{P}^2(\Lambda)$ have this form, we just have to see that $\Xi_\Lambda F_i$ preserves non-injectivity. From [8](19.10), the equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$ induces an isomorphism of bifunctors $\mathrm{Ext}_{\mathcal{D}}(-, ?) \cong \mathrm{Ext}_{\mathcal{P}^1(\Lambda)}(\Xi_\Lambda(-), \Xi_\Lambda(?))$. Hence, we have to see that F_i preserves non-injectivity. Whenever $\mathcal{A}$ is an admissible ditalgebra, the reduction functors $F^z \in \{F^X, F^d, F^r\}$ are exact and induce injective morphisms $\mathrm{Ext}_{\mathcal{A}^z}(-, ?) \longrightarrow \mathrm{Ext}_{\mathcal{A}}(F^z(-), F^z(?))$, see [8](16.6), (9.6), (9.9) and (9.10), hence they preserve non-injectivity. Then, the finite composition G of them has the same property. It is also clear that the functor $B \otimes_B -$ preserves non-injectivity. The functor E also has this property, as a consequence of (8.1). This finishes the proof of Claim 2. □

Now, for $i \in [1, m]$, set $Z_i := Z \otimes_D F_i(B_i)$, where Z is the transition bimodule associated to Λ, as in [8](22.18). Then, each Z_i is finitely generated over B_i by construction.

For each i, denote by H_i the composition

$$B_i\text{-Mod} \xrightarrow{F_i} \mathcal{D}^{\Lambda}\text{-Mod} \xrightarrow{\Xi_\Lambda} P^1(\Lambda) \xrightarrow{\text{Cok}} \Lambda\text{-Mod},$$

which is, by [8](22.18), naturally isomorphic to

$$\text{Cok}\, \Xi_\Lambda(F_i(B_i) \otimes_{B_i} -) \cong Z \otimes_D F_i(B_i) \otimes_{B_i} - = Z_i \otimes_{B_i} -.$$

(1) From Claim 2 and [8](18.10), if B_i is of the first type, H_i preserves indecomposables which are non-injective. If $M, N \in B_i$-Mod have no injective direct summand, then $\Xi_\Lambda F_i(M), \Xi_\Lambda F_i(N)$ also have this property (use [8](29.4) and the fact that F_i preserves non-injectivity). Thus, $H_i(M) \cong H_i(N)$ implies that $M \cong N$.

(2) From Claim 1 and [8](18.10), if B_i is of the second type, H_i preserves indecomposables and isomorphism classes. Thus, item 2 holds.

(3) Let M be an indecomposable Λ-module with $\dim_k M \leq d$ and consider $L \in \mathcal{D}$-Mod with $\text{Cok}\, \Xi_\Lambda(L) \cong M$. From (4.4)(2), we get $\dim_k L \leq d'$. From (7.6), we know that for almost all such modules L, we have that $L \cong F_i(N)$, for some $i \in [1, m]$ and $N \in B_i$-Mod. Hence, $M \cong \text{Cok}\, \Xi_\Lambda(L) \cong \text{Cok}\, \Xi_\Lambda F_i(N) \cong Z_i \otimes_{B_i} N$.

(4) Assume that $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $\text{Cok}\, \Xi_\Lambda F_i(N_u) \cong \text{Cok}\, \Xi_\Lambda F_j(M_u)$, for all $u \in U$. The families contain no injective indecomposable B_i-module. Then, $\Xi_\Lambda F_i(N_u), \Xi_\Lambda F_j(M_u) \in \mathcal{P}^2(\Lambda)$ and the existence of an isomorphism $\text{Cok}\, \Xi_\Lambda F_i(N_u) \cong \text{Cok}\, \Xi_\Lambda F_j(M_u)$ in Λ-Mod, together with [8](18.10)(3), imply that $F_i(N_u) \cong F_j(M_u)$, for $u \in U$. From (7.6)(3), we get $i = j$. □

9. Almost split sequences and generic tameness

Lemma 9.1. *Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are admissible ditalgebras over any field k. Consider the corresponding usual exact structures on their categories of modules; see* [8](6.8). *Assume that $F : \mathcal{A}'\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is a fully faithful functor, that we have a pair of composable morphisms $M \xrightarrow{f} L \xrightarrow{g} N$ in $\mathcal{A}'$-Mod such that*

$$F(M) \xrightarrow{F(f)} F(L) \xrightarrow{F(g)} F(N)$$

is an almost split conflation in $\mathcal{A}$-Mod, then $M \xrightarrow{f} L \xrightarrow{g} N$ is an almost split conflation in $\mathcal{A}'$-Mod.

Proof. Since $F(gf) = F(g)F(f) = 0$, we have that $gf = 0$. It is also easy to see, using that F is fully faithful, that $f = \text{Ker}\, g$ and $g = \text{Coker}\, f$ follow from $F(f) = \text{Ker}\, F(g)$ and $F(g) = \text{Coker} F(f)$. Since in the layer (R', W') of $\mathcal{A}'$, the algebra R' is semisimple, we obtain that $M \xrightarrow{f} L \xrightarrow{g} N$ is a conflation (see [8](6.6)). Again, since $F(M)$ and $F(N)$ are indecomposable and F is fully faithful, we have that M and N are indecomposable. It is also straightforward to verify, using that F is full and faithful, that g is right almost split and f is left almost split. □

Lemma 9.2. *Assume that B is an initial subalgebra of the admissible ditalgebra $\mathcal{A}$ over any field k. Consider the associated restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow B\text{-Mod}$ and the extension functor $E : B\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Suppose that we are given an almost split conflation in $\mathcal{A}$-mod*

$$\xi \quad : \quad E(M) \quad \xrightarrow{u} \quad L \quad \xrightarrow{v} \quad E(N)$$

where M, N are some finite-dimensional indecomposable B-modules. Then, the following holds,

1. *Whenever the restricted sequence $R(\xi)$ splits in B-mod, the module L is indecomposable.*
2. *Whenever the restricted sequence $R(\xi)$ does not split in B-mod, it is an almost split sequence and, moreover, $ER(\xi) \cong \xi$.*

Proof. (1) Suppose that the sequence $R(\xi)$ splits in B-mod and assume that L is not indecomposable. Then $L \cong L_1 \oplus L_2$ for some non-trivial modules $L_1, L_2 \in \mathcal{A}$-mod, and $R(L_1) \oplus R(L_2) \cong R(L) \cong M \oplus N$. Thus, the B-modules $R(L_1), M, N, R(L_2)$ are indecomposables. The morphism u has the matrix form $u = [u_1, u_2]^t$, where $u_1 : E(M) \longrightarrow L_1$ and $u_2 : E(M) \longrightarrow L_2$. Since $R(\xi)$ splits, the morphism $R(u) = [R(u_1), R(u_2)]^t$ is a section and, hence, there is a morphism of B-modules $[\mu_1, \mu_2] : R(L_1) \oplus R(L_2) \longrightarrow M$ such that $\mu_1 R(u_1) + \mu_2 R(u_2) = 1_M$. But, $\mathrm{End}_B(M)$ is a local algebra, so $\mu_1 R(u_1)$ or $\mu_2 R(u_2)$ is an isomorphism. Assume that the first one is an isomorphism and write down the components of the morphism $u_1 = (u_1^0, u_1^1) \in \mathrm{Hom}_{\mathcal{A}}(E(M), L_1)$. Then, the composition of $R(u_1) = u_1^0 : M \longrightarrow R(L_1)$ with $\mu_1 : R(L_1) \longrightarrow M$ is an isomorphism. Thus, u_1^0 is a section in B-mod between indecomposables, hence it is an isomorphism. Therefore, $u_1 : E(M) \longrightarrow L_1$ is an isomorphism in $\mathcal{A}$-mod, and this implies that the conflation ξ splits; a contradiction. Thus, L is indecomposable.

(2) Assume that the sequence $R(\xi)$ does not split. Let us show that it is an almost split sequence in B-mod. Take $Z \in B$-mod and a non-retraction $f \in \mathrm{Hom}_B(Z, N)$. Then, we have the non-retraction $E(f) \in \mathrm{Hom}_{\mathcal{A}}(E(Z), E(N))$ and, since ξ is an almost split sequence, there is a morphism $t \in \mathrm{Hom}_{\mathcal{A}}(E(Z), L)$ such that $vt = E(f)$. Apply R to obtain $R(v)R(t) = RE(f) = f$, and that $R(\xi)$ is an almost split sequence. Now, consider the non-split conflation obtained from $R(\xi)$ by applying E, and compare it with the original almost split conflation ξ to derive a commutative diagram

$$\begin{tikzcd}
ER(\xi) \arrow[r, phantom, ":"] & E(M) \arrow[r, "ER(u)"] \arrow[d, "h'"] & ER(L) \arrow[r, "ER(v)"] \arrow[d, "h"] & E(N) \arrow[d, equal] \\
\xi \arrow[r, phantom, ":"] & E(M) \arrow[r, "u"] & L \arrow[r, "v"] & E(N).
\end{tikzcd}$$

Since $R(v)R(h) = RER(v) = R(v)$ and $R(v)$ is a minimal right almost split morphism, we obtain that $R(h)$ is an isomorphism. It follows that h is an isomorphism in $\mathcal{A}$-mod, and so is h'. □

Proposition 9.3. *Assume that B is an initial subalgebra of the admissible ditalgebra $\mathcal{A}$ over the field k. Consider the associated restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow B\text{-Mod}$ and the extension functor $E : B\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Consider a finite-dimensional admissible ditalgebra $\mathcal{D}$, as in* [8](7.15)*, and a full and faithful exact functor $F : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$. Suppose that there is a sequence $\{N_n\}_{n \in \mathbb{N}}$ of finite-dimensional indecomposable B-modules and almost split sequences in B-mod*

$$\zeta_1 : N_1 \longrightarrow N_2 \longrightarrow N_1,$$
$$\zeta_n : N_n \longrightarrow N_{n+1} \oplus N_{n-1} \longrightarrow N_n, \quad \textit{for } n \geq 2.$$

Then, we have the following.

1. *If* $FE(\zeta_1)$ *is an almost split conflation in* $\mathcal{D}$-mod*, all the images* $\{E(\zeta_n)\}_{n\in\mathbb{N}}$ *are almost split conflations in* $\mathcal{A}$-mod*, and all the images* $\{FE(\zeta_n)\}_{n\in\mathbb{N}}$ *are almost split conflations in* $\mathcal{D}$-mod*.*
2. *If* $E(\zeta_1)$ *is an almost split conflation in* $\mathcal{A}$-mod *and* $FE(N_1)$ *is homogeneous, then* $FE(\zeta_1)$ *is an almost split conflation in* $\mathcal{D}$-mod*.*

Proof. From [8](9.6) and (8.1), the exact functors E and F induce injections of bifunctors $\mathrm{Ext}_{\mathcal{B}}(-,?) \longrightarrow \mathrm{Ext}_{\mathcal{A}}(E(-),E(?)) \longrightarrow \mathrm{Ext}_{\mathcal{D}}(FE(-),FE(?))$ and, hence, $E(N_n)$ is not projective in the exact category $\mathcal{A}$-mod and $FE(N_n)$ is not projective in the exact category $\mathcal{D}$-mod, for all $n\in\mathbb{N}$.

(1) By assumption the conflation $E(\zeta_1): E(N_1) \longrightarrow E(N_2) \longrightarrow E(N_1)$ is mapped by the full and faithful functor F to the almost split conflation $FE(\zeta_1)$. From (9.1), $E(\zeta_1)$ is an almost split conflation in $\mathcal{A}$-mod.

Since $\mathcal{D}$ is a finite-dimensional ditalgebra, the category $\mathcal{D}$-mod has almost split conflations; see [8](7.18) and [8](5.6). Given an almost split conflation $M \longrightarrow L \longrightarrow N$ in $\mathcal{D}$-mod, we use the notation $\tau N := M$.

Let us work first on the case $n = 2$. Assume $Z_2 \longrightarrow M_2 \longrightarrow FE(N_2)$ is an almost split conflation in $\mathcal{D}$-mod. Since $FE(\zeta_1)$ is an almost split conflation, there is an irreducible morphism $FE(N_1) \longrightarrow FE(N_2)$. Then, $FE(N_1)$ is isomorphic to a direct summand of the middle term M_2, and there is an irreducible morphism $Z_2 \longrightarrow FE(N_1)$. Then, $Z_2 \cong FE(N_2)$ and we have an almost split conflation $\xi_2 : FE(N_2) \longrightarrow M_2 \longrightarrow FE(N_2)$. Now, $E(\zeta_2)$ is a non-split conflation in $\mathrm{Ext}_{\mathcal{A}}(E(N_2),E(N_2))$ and this module is embedded as a submodule of $\mathrm{Ext}_{\mathcal{D}}(FE(N_2),FE(N_2))$ over the endomorphism algebra, which implies that there is an almost split conflation $\zeta_2' : E(N_2) \longrightarrow M_2' \longrightarrow E(N_2)$ in $\mathcal{A}$-mod such that $F(\zeta_2') \cong \xi_2$. Since, $E(\zeta_1)$ is an almost split conflation, the indecomposable $E(N_1)$ is isomorphic to a direct summand of M_2'. But if M_2' was indecomposable, we would have $M_2' \cong E(N_1)$ and this is impossible because $\dim_k M_2' = 2\dim_k E(N_2) = 4\dim_k E(N_1)$. Thus, M_2' is decomposable and we can apply (9.2) to the conflation ζ_2' to obtain that $R(\zeta_2')$ is an almost split sequence (hence $R(\zeta_2') \cong \zeta_2$), and $\zeta_2' \cong ER(\zeta_2') \cong E(\zeta_2)$ is an almost split conflation in $\mathcal{A}$-mod, and $FE(\zeta_2) \cong F(\zeta_2') \cong \xi_2$ is an almost split conflation.

Now, we proceed by induction on n. Assume $n \geq 2$ and that we already know that $E(\zeta_n)$ and $E(\zeta_{n-1})$ are almost split conflations in $\mathcal{A}$-mod and that $FE(\zeta_n)$ and $FE(\zeta_{n-1})$ are almost split conflations in $\mathcal{D}$-mod. Consider the almost split conflation $Z_{n+1} \longrightarrow M_{n+1} \longrightarrow FE(N_{n+1})$ in $\mathcal{D}$-mod. Since $FE(\zeta_n)$ is an almost split conflation, there is an irreducible morphism $FE(N_n) \longrightarrow FE(N_{n+1})$. Then, $FE(N_n)$ is isomorphic to a direct summand of the middle term M_{n+1}, and there is an irreducible morphism $Z_{n+1} \longrightarrow FE(N_n)$. Then, $Z_{n+1} \cong FE(N_{n+1})$ or $Z_{n+1} \cong FE(N_{n-1})$. The second case is excluded because $FE(\zeta_{n-1})$ is an almost split conflation ending at $FE(N_{n-1}) \ncong FE(N_{n+1})$ (since $N_{n-1} \ncong N_{n+1}$). Then, $Z_{n+1} \cong FE(N_{n+1})$ and we have an almost split conflation $\xi_{n+1} : FE(N_{n+1}) \longrightarrow M_{n+1} \longrightarrow FE(N_{n+1})$. Now, $E(\zeta_{n+1})$ is a non-split conflation in $\mathrm{Ext}_{\mathcal{A}}(E(N_{n+1}),E(N_{n+1}))$ and this module is embedded as a submodule of $\mathrm{Ext}_{\mathcal{D}}(FE(N_{n+1}),FE(N_{n+1}))$ over the endomorphism algebra, which implies that there is an almost split conflation $\zeta_{n+1}' : E(N_{n+1}) \longrightarrow M_{n+1}' \longrightarrow E(N_{n+1})$ in $\mathcal{A}$-mod such that $F(\zeta_{n+1}') \cong \xi_{n+1}$. Since, $E(\zeta_n)$ is an almost split conflation, the indecomposable $E(N_n)$ is isomorphic to a direct summand of M_{n+1}'. But if M_{n+1}' was indecomposable, we would have $M_{n+1}' \cong E(N_n)$ and this is impossible because $\dim_k M_{n+1}' = 2\dim_k E(N_{n+1}) > \dim_k E(N_n)$. Here we used the inequality $\dim_k E(N_{n+1}) > \dim_k E(N_n)$, which can be proved by induction on n, using the given exact sequences $\{\zeta_n\}_{n\in\mathbb{N}}$. Thus, M_{n+1}' is decomposable and we can apply (9.2) to the conflation ζ'_{n+1} to obtain that $R(\zeta'_{n+1})$ is an almost split sequence (hence $R(\zeta'_{n+1}) \cong \zeta_{n+1}$), and $\zeta'_{n+1} \cong ER(\zeta'_{n+1}) \cong E(\zeta_{n+1})$ is an almost split conflation in $\mathcal{A}$-mod, and $FE(\zeta_{n+1}) \cong F(\zeta'_{n+1}) \cong \xi_{n+1}$ is an almost split conflation in $\mathcal{D}$-mod. Our induction is now complete.

(2) Observe that if $\zeta : X \longrightarrow L \longrightarrow Y$ is a non-split conflation in $\mathcal{A}$-mod then $F(\zeta)$ is a non-split conflation of $\mathcal{D}$-mod, because of the observation preceding the proof of (1). Now, if $\zeta : X \longrightarrow L \longrightarrow Y$ is an almost split conflation in $\mathcal{A}$-mod and we have an isomorphism $h : F(X) \longrightarrow \tau F(Y)$, then $F(\zeta)$ is an almost split conflation in $\mathcal{D}$-mod. Indeed, $\zeta \in \mathrm{socExt}_{\mathcal{A}}(Y, X)$, where $\mathrm{Ext}_{\mathcal{A}}(Y, X)$ is considered as a right $\mathrm{End}_{\mathcal{A}}(Y)$-module. Since F is exact, full and faithful, $F(\zeta) \in \mathrm{socExt}_{\mathcal{D}}(F(Y), F(X))$, where $\mathrm{Ext}_{\mathcal{D}}(F(Y), F(X))$ is considered as a right $\mathrm{End}_{\mathcal{D}}(F(Y))$-module. But then, the isomorphism

$$h^* : \mathrm{Ext}_{\mathcal{D}}(F(Y), F(X)) \longrightarrow \mathrm{Ext}_{\mathcal{D}}(F(Y), \tau F(Y))$$

maps $F(\zeta)$ onto a conflation $\widehat{\zeta}$ in the simple socle of $\mathrm{Ext}_{\mathcal{D}}(F(Y), \tau F(Y))$; see [8](7.18). Therefore, the isomorphic conflations $F(\zeta)$ and $\widehat{\zeta}$ are almost split conflations in $\mathcal{D}$-mod. Apply this argument to $\zeta = \zeta_1$ to obtain that $F(\zeta_1) : F(N_1) \longrightarrow F(N_2) \longrightarrow F(N_1)$ is an almost split conflation in $\mathcal{D}$-mod. □

Proposition 9.4. *Assume that the admissible ditalgebra $\mathcal{A}$ is constructible from the generically tame finite-dimensional basic algebra Λ, over the infinite perfect field k. Then, for any $b \in \mathbb{N}$, almost every indecomposable $\mathcal{A}$-module M with* $\dim_k M \leq b$ *and* $\mathrm{Ext}_{\mathcal{A}}(M, M) \neq 0$ *admits an almost split conflation in* $\mathcal{A}$-mod *of the form $M \longrightarrow L \longrightarrow M$.*

Proof. Let us call a ditalgebra $\mathcal{B}$ *almost homogeneous* iff, for any $b \in \mathbb{N}$, almost every indecomposable $\mathcal{A}$-module M with $\dim_k M \leq b$ admits an almost split conflation in $\mathcal{A}$-mod of the form $M \longrightarrow L \longrightarrow M$.

Assume that $\mathcal{A}$ is constructible from the generically tame finite-dimensional basic algebra Λ. Then, with the notation of (4.2) in mind, we have an isomorphism of layered ditalgebras $\xi : \mathcal{D}^{z_1 \cdots z_t} \longrightarrow \mathcal{A}$ and the functors

$$\mathcal{A}\text{-Mod} \xrightarrow{F} \mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\mathrm{Cok}} \Lambda\text{-Mod},$$

where $F = F^{z_1} \cdots F^{z_t} F_\xi$ is the composition of the corresponding reduction functors $F^{z_i} : \mathcal{D}^{z_1 \cdots z_i}\text{-Mod} \longrightarrow \mathcal{D}^{z_1 \cdots z_{i-1}}\text{-Mod}$, for $i \in [1, t]$, and F_ξ.

We will show first that the Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$ is almost homogeneous. For this, we use that the ground field is infinite and perfect, thus, from [23](5.3), the algebra Λ is almost homogeneous.

Fix $b \in \mathbb{N}$. Assume that $N \in \mathcal{D}$-mod satisfies that $\dim_k N \leq b$ and $\Xi_\Lambda(N) \in \mathcal{P}^2(\Lambda)$. Make $M := \mathrm{Cok}\, \Xi_\Lambda(N)$, then from (4.4)(1), we get $\dim_k M \leq \dim_k \Lambda \times b$. Make $b' := \dim_k \Lambda \times b$. Since Λ is almost homogeneous, for almost every indecomposable $M \in \Lambda$-mod with $\dim_k M \leq b'$, we have an almost split sequence of the form $M \longrightarrow L \longrightarrow M$.

From [8](7.18) and [8](19.5), we know that $\mathcal{D}$-mod has almost split conflations. From [8](19.10), [8](18.12) and [8](18.13), for almost every (non-projective) indecomposable $N \in \mathcal{D}$-mod with $\dim_k N \leq b$, the almost split conflation $N' \longrightarrow L \longrightarrow N$ in $\mathcal{D}$-mod is mapped onto an almost split sequence

$$\mathrm{Cok}\, \Xi_\Lambda(N') \longrightarrow \mathrm{Cok}\, \Xi_\Lambda(L) \longrightarrow \mathrm{Cok}\, \Xi_\Lambda(N)$$

in Λ-mod. But $\dim_k \mathrm{Cok}\, \Xi_\Lambda(N) \leq b'$, thus, almost for every such N, we have $\mathrm{Cok}\, \Xi_\Lambda(N') \cong \mathrm{Cok}\, \Xi_\Lambda(N)$. Since N' is not injective, the object $\Xi_\Lambda(N')$ lies in $\mathcal{P}^2(\Lambda)$ and, from [8](18.10), $\Xi_\Lambda(N') \cong \Xi_\Lambda(N)$. Thus, $N' \cong N$. We have shown that $\mathcal{D}$ is almost homogeneous.

Now, recall that the functor F is exact full and faithful, and that it induces an injective morphism $\mathrm{Ext}_{\mathcal{A}}(-,?) \longrightarrow \mathrm{Ext}_{\mathcal{D}}(F(-), F(?))$. Moreover, there is a constant C such that $\dim_k F(N) \leq C \times \dim_k N$, for any $N \in \mathcal{A}$-Mod. Now, fix $b \in \mathbb{N}$. Then, for $b' := C \times b$, we know that for almost all $M \in \mathcal{D}$-mod with $\dim_k M \leq b'$ there is an almost split conflation $M \longrightarrow L \longrightarrow M$. Recall that, for any $N \in \mathcal{A}$-mod with $\dim_k N \leq b$, the functor F induces an injective morphism of $\mathrm{End}_{\mathcal{A}}(N)$-modules

$$\mathrm{Ext}_{\mathcal{A}}(N, N) \xrightarrow{F^*} \mathrm{Ext}_{\mathcal{D}}(F(N), F(N)).$$

Moreover, for almost all N, the $\mathrm{End}_{\mathcal{A}}(N)$-module $\mathrm{Ext}_{\mathcal{D}}(F(N), F(N))$ has simple socle, which is generated by an almost split conflation $\zeta \in \mathrm{Ext}_{\mathcal{D}}(F(N), F(N))$; see [8](7.19). Then, the non-zero module $\mathrm{Ext}_{\mathcal{A}}(N, N)$ has also simple socle and the restriction of F^* to the socles of both modules is an isomorphism. In particular, there is a conflation $\xi : N \longrightarrow L \longrightarrow N$ in $\mathrm{Ext}_{\mathcal{A}}(N, N)$ such that $F(\xi) = \zeta$. Then, we can apply (9.1) to guarantee that ξ is an almost split conflation of $\mathcal{A}$-mod. This ends our proof. $\square$

Proposition 9.5. *Assume that $\mathcal{B}$ is an initial subalgebra of the admissible ditalgebra $\mathcal{A}$, over the infinite perfect field k. Assume that $\mathcal{A}$ is constructible from the generically tame finite-dimensional basic algebra Λ. Consider the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Furthermore, suppose that, for each p in some index set P, there is a sequence $\{E_n^p\}_{n\in\mathbb{N}}$ of pairwise non-isomorphic finite-dimensional indecomposable B-modules and almost split sequences*

$$\zeta_1^p : E_1^p \longrightarrow E_2^p \longrightarrow E_1^p,$$
$$\zeta_n^p : E_n^p \longrightarrow E_{n+1}^p \oplus E_{n-1}^p \longrightarrow E_n^p, \quad \text{for } n \geq 2,$$

and also that, for any $b \in \mathbb{N}$, almost every indecomposable B-module M with $\dim_k M \leq b$ is isomorphic to E_n^p, for some $p \in P$ and $n \in \mathbb{N}$. Then, for any $d \in \mathbb{N}$ and almost every indecomposable $N \in B$-Mod with $\dim_k N \leq d$, the extension functor E maps almost split sequences with final term N onto almost split conflations.

Proof. Let us denote by $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$ the corresponding restriction functor and fix $d \in \mathbb{N}$. Then, from (5.2), there is a finite family $\mathcal{I}(2d)$ of indecomposable $\mathcal{B}$-modules such that, for any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq 2d$ and $M \ncong E(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{B}$-Mod, the module $R(M)$ is isomorphic in $\mathcal{B}$-Mod to a direct sum of modules in $\mathcal{I}(2d)$.

We have already seen in (8.1) that E is an exact functor which preserves indecomposables and non-split exact sequences.

By assumption, we have a family $\{\mathcal{T}_p\}_{p\in P}$ of tubes in the Auslander–Reiten graph of $\mathcal{B}$-mod, see [8](32.9), such that almost every indecomposable B-module N with $\dim_k N \leq d$ belongs to one of the tubes. Assume that $X_1, \ldots, X_t$ is a set of representatives of the indecomposable B-modules in $\mathcal{I}(2d)$. Since, $\dim_k E_{n+1}^p > \dim_k E_n^p$, for all $p \in P$ and $n \in \mathbb{N}$, we can eliminate all the tubes which include any of the modules X_i, and obtain a smaller family $\{\mathcal{T}_p\}_{p\in P'}$, such that no X_i belongs to any of these tubes and almost every indecomposable B-module N with $\dim_k N \leq d$ belongs to one of the tubes in this smaller family of tubes. Let us denote by $\mathcal{M}'_d$ the class of such indecomposable B-modules N, which satisfy that $\dim_k N \leq d$ and that they have an almost split sequence $N \longrightarrow L \longrightarrow N$ such that N and the indecomposable direct summands of L are not isomorphic to modules in $\mathcal{I}(2d)$. Thus, almost every indecomposable B-module N with $\dim_k N \leq d$ lies in $\mathcal{M}'_d$. By assumption, any $N \in \mathcal{M}'_d$ satisfies that $\mathrm{Ext}_{\mathcal{B}}(N, N) \neq 0$, thus also $\mathrm{Ext}_{\mathcal{A}}(E(N), E(N)) \neq 0$. Finally, consider the subclass $\mathcal{M}_d$ of $\mathcal{M}'_d$ obtained by the elimination of all $N \in \mathcal{M}'_d$ with $E(N)$ not homogeneous in $\mathcal{A}$-mod. Almost every module in $\mathcal{M}'_d$ is in $\mathcal{M}_d$, by (9.4). Our final goal is to show that E preserves almost split sequences ending at any $N \in \mathcal{M}_d$.

We first show that, given $N \in \mathcal{M}_d$, if we consider the almost split conflation in $\mathcal{A}$-mod,

$$\xi : E(N) \xrightarrow{u} L \xrightarrow{v} E(N)$$

then, its restriction $R(\xi) : N \xrightarrow{u^0} R(L) \xrightarrow{v^0} N$ is an almost split sequence in B-mod. Indeed, assume first that $R(\xi)$ splits. Then, from (9.2), we know that L is indecomposable. Also, $R(L) \cong N \oplus N$ is not isomorphic to a direct sum of B-modules in $\mathcal{I}(2d)$. Since $\dim_k L = 2\dim_k N \leq 2d$, by the first paragraph of this proof, we get $L \cong E(H)$, for some $H \in B$-Mod. Hence $H \cong RE(H) \cong R(L) \cong N \oplus N$ and $L \cong E(H) \cong E(N) \oplus E(N)$, a contradiction. Then, $R(\xi)$ does not split and, again from (9.2), $R(\xi)$ is an almost split sequence and $ER(\xi) \cong \xi$. Then, we may assume that in the chosen almost split conflation ξ ending at $E(N)$, the middle term has the form $L = E(H)$, for some $H \in B$-mod.

Finally, assume that $N \in \mathcal{M}_d$ and that $\theta : N \xrightarrow{f} Z \xrightarrow{g} N$ is an almost split sequence in B-mod. We shall prove that

$$E(\theta) : E(N) \xrightarrow{E(f)} E(Z) \xrightarrow{E(g)} E(N)$$

is an almost split conflation in $\mathcal{A}$-mod. Indeed, we know that $E(\theta)$ does not split by (8.1). From the previous considerations, we have an almost split sequence

$$\xi : E(N) \xrightarrow{u} E(H) \xrightarrow{v} E(N)$$

in $\mathcal{A}$-mod. Then, comparing this sequence with the extension of the previous one, we obtain the commutative diagram

$$\begin{array}{ccccccc}
E(\theta) & : & E(N) & \xrightarrow{E(f)} & E(Z) & \xrightarrow{E(g)} & E(N) \\
 & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Id} \\
\xi & : & E(N) & \xrightarrow{u} & E(H) & \xrightarrow{v} & E(N).
\end{array}$$

Then, apply the functor R and compare $R(\xi)$ with the original sequence θ to obtain the commutative diagram

$$\begin{array}{ccccccc}
\theta & : & N & \xrightarrow{f} & Z & \xrightarrow{g} & N \\
 & & \downarrow{\scriptstyle t^0} & & \downarrow{\scriptstyle h^0} & & \downarrow{\scriptstyle Id} \\
R(\xi) & : & N & \xrightarrow{u^0} & H & \xrightarrow{v^0} & N \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle Id} \\
\theta & : & N & \xrightarrow{f} & Z & \xrightarrow{g} & N.
\end{array}$$

Since g is minimal right almost split, rh^0 is an isomorphism and h^0 is a section. Similarly, v^0 is minimal right almost split, thus h^0r is an isomorphism and h^0 is a retraction. Hence, h^0 is an isomorphism. It follows that h is an isomorphism, as well as t. Thus, $E(\theta) \cong \xi$ is an almost split conflation. $\square$

Corollary 9.6. *Assume that $\mathcal{B}$ is an initial subalgebra of the admissible ditalgebra $\mathcal{A}$, over the infinite perfect field k. Assume that $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra Λ. Consider the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, Drozd's ditalgebra $\mathcal{D}$ of Λ and the associated composition of reduction functors $F : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$. Furthermore, suppose that, for each p in some index set P, there is a sequence $\{N_n^p\}_{n\in\mathbb{N}}$ of non-isomorphic finite-dimensional indecomposable B-modules and almost split sequences*

$$\zeta_1^p : N_1^p \longrightarrow N_2^p \longrightarrow N_1^p,$$
$$\zeta_n^p : N_n^p \longrightarrow N_{n+1}^p \oplus N_{n-1}^p \longrightarrow N_n^p, \quad \text{for } n \geq 2,$$

and also that, for any $b \in \mathbb{N}$, almost every indecomposable B-module M with $\dim_k M \leq b$ is isomorphic to N_n^p, for some $p \in P$ and $n \in \mathbb{N}$. For each $d \in \mathbb{N}$, define $P_d := \{p \in P \mid \dim_k N_1^p \leq d\}$. Then, for each $d \in \mathbb{N}$, almost every $p \in P_d$ determines the family of almost split conflations $\{FE(\zeta_n^p)\}_{n\in\mathbb{N}}$.

Proof. From (9.5), we know that for each $d \in \mathbb{N}$, there is a (possibly empty) finite subset Q_d of P_d, such that $E(\zeta_1^p)$ is an almost split conflation for $p \in P_d \setminus Q_d$.

From (7.3), we know that there is a constant C such that $\dim_k FE(N) \leq C \dim_k N$, for all $N \in \mathcal{B}\text{-mod}$. Thus, for $p \in P_d$, we get $\dim_k FE(N_1^p) \leq C \dim_k N_1^p \leq Cd$. In the proof of (9.4), we showed that for each natural number almost every $\mathcal{D}$-module with dimension bounded by this natural number is homogeneous. It follows that there is a (possibly empty) finite subset Q'_d of P_d, such that $FE(N_1^p)$ is homogeneous for $p \in P_d \setminus Q'_d$.

Then, for all $p \in P \setminus Q_d \cup Q'_d$, the module $FE(N_1^p)$ is homogeneous and $E(\zeta_1^p)$ is an almost split conflation in $\mathcal{D}$-mod. Then, we can apply (9.3) to finish the proof. □

10. Parametrizations over principal ideal domains

Proposition 10.1. *Let Λ be a generically tame finite-dimensional basic algebra over the infinite perfect field k. Consider the associated Drozd's ditalgebra $\mathcal{D}$. Then, for any $d \in \mathbb{N}$, there are constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and generically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and bounded principal ideal domains $\Gamma_1, \ldots, \Gamma_m$, where each Γ_i is associated to B_i as in (6.8), and a family of functors $\hat{F}_1, \ldots, \hat{F}_m$ satisfying the following.*

1. *Each functor $\hat{F}_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$ preserves indecomposability and isomorphism classes, for any $i \in [1, m]$.*
2. *For almost every indecomposable $M \in \mathcal{D}\text{-Mod}$ with dimension $\leq d$ there exist $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$ such that $\hat{F}_i(N) \cong M$ in $\mathcal{D}\text{-Mod}$.*
3. *If $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in $\Gamma_i\text{-mod}$ and $\Gamma_j\text{-mod}$, respectively, such that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$ for all $u \in U$, then $i = j$.*
4. *Each functor $\hat{F}_i$ is given by the following composition:*

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{G_i} \mathcal{D}\text{-Mod},$$

where $H_i : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$ is the functor of (6.8), E_i is the associated extension functor and G_i is the composition of the reduction functors associated to the finite sequence of reductions which transform $\mathcal{D}$ into $\mathcal{A}_i$.

5. *For any $i \in [1, m]$, adopt the notation of (6.5) with $\Gamma = \Gamma_i$; for $b \in \mathbb{N}$, denote $P_b := \{p \in P \mid \dim_k E_1^p \leq b\}$; then, for any $b \in \mathbb{N}$, almost every $p \in P_b$ determines almost split conflations*

$$\hat{F}_i(\zeta_1^p) : \hat{F}_i E_1^p \longrightarrow \hat{F}_i E_2^p \longrightarrow \hat{F}_i E_1^p \quad \text{and, for } n \geq 2,$$
$$\hat{F}_i(\zeta_n^p) : \hat{F}_i E_n^p \longrightarrow \hat{F}_i E_{n+1}^p \oplus \hat{F}_i E_{n-1}^p \longrightarrow \hat{F}_i E_n^p.$$

Proof. Fix a natural number d. We already know that $\mathcal{D}$-mod has almost split conflations and, from (9.4), for every natural number d, almost every indecomposable $\mathcal{D}$-module M with $\dim_k M \leq d$ is homogeneous.

From (7.6), applied to $\mathcal{D}$, we know the existence of the families of constructible ditalgebras $\{\mathcal{A}_i\}_i$, of initial generically tame minimal subalgebras $\{B_i\}_i$, and of functors $F_i := G_i E_i : B_i\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$. From (6.8), we have the existence of the bounded principal ideal domains $\{\Gamma_i\}_i$ and the functors $H_i : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$. Then, we can define the functors $\hat{F}_i$ as proposed in item (4).

(1) From (7.6)(1), each F_i preserves indecomposability and isomorphism classes. This is also the case for the full and faithful functors H_i. Then (1) is clear.

(2) For almost every $\mathcal{D}$-module M with dimension $\leq d$, there is $N \in B_i$-mod with $F_i(N) \cong M$. From (6.8), we know that H_i covers almost every indecomposable B_i-module with bounded dimension. Then, after the elimination of all those $\mathcal{D}$-modules of the form $F_i(N)$ where N is not covered by H_i, we remain with almost all $\mathcal{D}$-modules with dimension bounded by d.

(3) Assume that there are infinite families $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ of pairwise non-isomorphic indecomposable modules in Γ_i-mod and Γ_j-mod, respectively, such that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$, for all $u \in U$. Then, we have infinite families of pairwise non-isomorphic indecomposable regular modules $\{H_i N_u\}_{u\in U}$ and $\{H_j M_u\}_{u\in U}$ in B_i-mod and B_j-mod, respectively, such that $F_i(H_i N_u) \cong F_j(H_j M_u)$, for all $u \in U$. From, (7.6)(3), we obtain $i = j$.

(5) Fix $i \in [1, m]$ and apply (6.8) to H_i to obtain the almost split sequences $H_i(\zeta_n^p)$ in B_i-mod, for $n \in \mathbb{N}$. Then, we can apply (9.6) to B_i and $\mathcal{A}_i$. $\square$

Theorem 10.2. *Let Λ be a generically tame finite-dimensional algebra over an infinite perfect field k and let d be a non-negative integer. Then, there is a finite sequence of bounded principal ideal domains $\Gamma_1, \ldots, \Gamma_m$ and Λ–Γ_i-bimodules $\hat{Z}_1, \ldots, \hat{Z}_m$, which are finitely generated as right Γ_i-modules, satisfying the following.*

1. *The functor $\hat{Z}_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes.*
2. *Almost every indecomposable Λ-module M with $\dim_k M \leq d$ is isomorphic to $\hat{Z}_i \otimes_{\Gamma_i} N$, for some $i \in [1, m]$ and some $N \in \Gamma_i$-mod.*
3. *If $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in Γ_i-mod and Γ_j-mod, respectively, such that $\hat{Z}_i \otimes_{\Gamma_i} N_u \cong \hat{Z}_j \otimes_{\Gamma_j} M_u$ for all $u \in U$, then $i = j$.*
4. *For any $i \in [1, m]$, adopt the notation of (6.5) with $\Gamma = \Gamma_i$; for $b \in \mathbb{N}$, make $P_b := \{p \in P \mid \dim_k E_1^p \leq b\}$; then, for all $b \in \mathbb{N}$, almost every $p \in P_b$ determines almost split conflations in Λ-mod*

$$\xi_1^p : \hat{Z}_i \otimes_{\Gamma_i} E_1^p \longrightarrow \hat{Z}_i \otimes_{\Gamma_i} E_2^p \longrightarrow \hat{Z}_i \otimes_{\Gamma_i} E_1^p \quad \text{and, for } n \geq 2,$$
$$\xi_n^p : \hat{Z}_i \otimes_{\Gamma_i} E_n^p \longrightarrow (\hat{Z}_i \otimes_{\Gamma_i} E_{n+1}^p) \oplus (\hat{Z}_i \otimes_{\Gamma_i} E_{n-1}^p) \longrightarrow \hat{Z}_i \otimes_{\Gamma_i} E_n^p.$$

Proof. We proceed as in the proof of (8.2). First observe that we can assume that Λ is a basic algebra. The argument to justify this is similar to the one used at the beginning of the proof of (8.2) and we skip it. Thus we assume that Λ is basic and, since our field is perfect, we can consider the associated Drozd's ditalgebra $\mathcal{D}$, the equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$ and the cokernel functor Cok $: \mathcal{P}^2(\Lambda) \longrightarrow \Lambda\text{-Mod}$.

Fix $d \in \mathbb{N}$ and apply last theorem to $d' := (1+\dim_k \Lambda)\dim_k \Lambda \times d$ to obtain the constructible ditalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m$, the pregenerically tame minimal algebras of infinite representation type $B_1, \dots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and the bounded principal ideal domains $\Gamma_1, \dots, \Gamma_m$, each Γ_i associated to B_i as in (6.8), and the family of functors $\hat{F}_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$ satisfying (10.1)(1)–(5) for $\mathcal{D}$ and d'.

(1) For a fixed $i \in [1, m]$, we claim that the functor $\Xi_\Lambda \hat{F}_i$ maps any indecomposable Γ_i-module N into $\mathcal{P}^2(\Lambda)$. Indeed, from [8](22.7), we have that

$$\Xi_\Lambda \hat{F}_i \cong \Xi_\Lambda G_i E_i H_i \cong \Xi_\Lambda L_{\mathcal{D}}(G_i E_i H_i(\Gamma_i) \otimes_{\Gamma_i} -).$$

Moreover, the Γ_i–Γ_i-bimodule Γ_i is mapped by H_i, which is a composition of Morita equivalences with restriction functors, onto a bimodule $H_i(\Gamma_i)$ which is finitely generated projective as a right Γ_i-module. This is also clear for the bimodule $E_i H_i(\Gamma_i)$. Then, from [8](22.7), this also holds for $Z_i := G_i E_i H_i(\Gamma_i)$. Then, from [8](22.20)(1), the module $\Xi_\Lambda \hat{F}_i(N)$ is not isomorphic to any object of the form $(Q, 0, 0)$ in $\mathcal{P}^1(\Lambda)$, and so $\Xi_\Lambda \hat{F}_i(N) \in \mathcal{P}^2(\Lambda)$. From [8](18.10), this implies that the following composition

$$\Gamma_i\text{-Mod} \xrightarrow{\hat{F}_i} \mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\text{Cok}} \Lambda\text{-Mod},$$

denoted by L_i, preserves indecomposables and isomorphism classes.

For $i \in [1, m]$, make $\hat{Z}_i := Z \otimes_D \hat{F}_i(\Gamma_i)$, where Z is the transition bimodule associated to Λ, as in [8](22.18). Then, each $\hat{Z}_i$ is finitely generated over Γ_i.

From [8](22.18), we have natural isomorphisms

$$L_i \cong \text{Cok}\, \Xi_\Lambda L_{\mathcal{D}}(\hat{F}_i(\Gamma_i) \otimes_{\Gamma_i} -) \cong Z \otimes_D \hat{F}_i(\Gamma_i) \otimes_{\Gamma_i} - = \hat{Z}_i \otimes_{\Gamma_i} -,$$

and our statement (1) follows.

(2) Let M be an indecomposable Λ-module with $\dim_k M \leq d$. From (4.4)(2), we get, for an indecomposable $L \in \mathcal{D}\text{-Mod}$ with $\text{Cok}\, \Xi_\Lambda(L) \cong M$, that $\dim_k L \leq d'$. Thus, from (10.1)(2), we have that $L \cong \hat{F}_i(N)$, for some $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$. Hence, $M \cong \text{Cok}\, \Xi_\Lambda(L) \cong \text{Cok}\, \Xi_\Lambda \hat{F}_i(N) \cong \hat{Z}_i \otimes_{\Gamma_i} N$, and (2) holds.

(3) Assume that $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposables in Γ_i-mod and Γ_j-mod, respectively, such that $\text{Cok}\, \Xi_\Lambda \hat{F}_i(N_u) \cong \text{Cok}\, \Xi_\Lambda \hat{F}_j(M_u)$, for all $u \in U$. As in the proof of item 1, using that Γ_i is a principal ideal domain, hence its finitely generated projective modules are free of finite rank, from [8](22.20), we obtain $\Xi_\Lambda \hat{F}_i(N_u), \Xi_\Lambda \hat{F}_j(M_u) \in \mathcal{P}^2(\Lambda)$. Then, [8](18.10)(3) implies that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$, for $u \in U$. Finally, apply (10.1)(3) to get $i = j$.

(4) This follows from (10.1)(5) and [8](18.13), after applying the functor $\text{Cok}\, \Xi_\Lambda$ to the family of almost split conflations $\hat{F}_i(\zeta_i^p)$. $\square$

Remark 10.3. In the context of the last theorem, as a consequence of item (4), we get: for all $p \in P$, with only a countable possible number of exceptions, we have the family of almost split conflations $\{\xi_n^p\}_{n\in\mathbb{N}}$ in Λ-mod. This is of course very weak if the field k is countable.

If k is algebraically closed (countable or not), we have $P = P_1$, because simple modules over rational k-algebras are one-dimensional, and we recover the situation studied by Crawley-Boevey in [13]. In case k is the field of real numbers, we encounter the possibility that some $\Gamma_i = \mathbb{R}[x]$, and here the simple Γ_i-modules have dimension bounded by 2, thus $P = P_2$, and again we have a nice situation where the whole families of almost split sequences $\{\xi_n^p\}_{n\in\mathbb{N}}$ appear in Λ-mod, for almost all prime $p \in \mathbb{R}[x]$. The situation in general seems to be more complex.

Remark 10.4. Let Λ be a generically tame algebra over the perfect infinite field k. Then, there are some important results on the Auslander–Reiten quiver of Λ which can be derived directly from their well known analogues for algebraically closed fields, using Kasjan's work. For instance, see the following.

1. Every connected component of the Auslander–Reiten quiver of Λ has at most a finite number of isoclasses of Λ-modules for each dimension.
2. For any $d \in \mathbb{N}$, almost every indecomposable Λ-module M with $\dim_k M \leq d$ is homogeneous.
3. For any $d \in \mathbb{N}$, almost every indecomposable Λ-module M with $\dim_k M \leq d$ lies in a homogeneous tube.

Indeed, to obtain (1): assume that the connected component $\mathcal{C}$ of the Auslander–Reiten quiver Γ of Λ contains Λ-modules $\{M_i\}_{i\in\mathbb{N}}$ with k-dimension $\leq d$. Denote by K the algebraic closure of k and notice that the extension $K : k$ is MacLane separable; see [22](3.1). Then, apply [22](4.4) to obtain connected components $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of the Auslander–Reiten quiver of Λ^K containing the indecomposable direct summands of all the Λ^K-modules $\{M_i^K\}_{i\in\mathbb{N}}$, all of them with K-dimension $\leq d$. Since the Λ-modules in the given family $\{M_i\}_{i\in\mathbb{N}}$ are non-isomorphic, their extensions $\{M_i^K\}_{i\in\mathbb{N}}$ are Λ^K-modules sharing no indecomposable direct summands, according to [22](2.5). Then, one of these components $\mathcal{D}_i$ contains an infinite number of Λ^K-modules with dimension bounded by d. This contradicts Crawley-Boevey's Corollary F in [13].

Statement (2) is just Kasjan's result [23](5.3), and statement (3) follows from the same argument used in the proof of Corollary E of [13].

These are very strong statements compared with the relative information we obtain in (10.2)(4). On the other hand, our result keeps track of how the original almost split sequences of the relevant principal ideal domains behave under the parametrizing functors. We stress the fact that Kasjan's (and of course Crawley-Boevey's) results play an essential role in the discussion of our Section 9.

Remark 10.5. The study of parametrizations of indecomposable modules of finite-dimensional algebras Λ over perfect fields is certainly not exhausted. The understanding of the relations between generic Λ-modules and families of indecomposables is not an easy problem. We have been able to make some progress for the case of real closed fields, which will appear in [7]. The case of finite fields, of central importance, seems to be quite more challenging.

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